A GENERALIZATION OF WATTS'S THEOREM: RIGHT EXACT FUNCTORS ON MODULE CATEGORIES

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Abstract. Watts's Theorem says that a right exact functor $F : \mathsf{Mod}R \to \mathsf{Mod}S$ that commutes with direct sums is isomorphic to $- \otimes_R B$ where B is the R-S-bimodule FR. The main result in this paper is the following: if A is a cocomplete category and $F : \mathsf{Mod}R \to \mathsf{A}$ is a right exact functor commuting with direct sums, then F is isomorphic to $- \otimes_R \mathcal{F}$ where $\mathcal{F}$ is a suitable R-module in A, i.e., a pair $(\mathcal{F}, \rho)$ consisting of an object $\mathcal{F} \in \mathsf{A}$ and a ring homomorphism $\rho : R \to \operatorname{Hom}_{\mathsf{A}}(\mathcal{F}, \mathcal{F})$. Part of the point is to give meaning to the notation $- \otimes_R \mathcal{F}$. That is done in the paper by Artin and Zhang [1] on Abstract Hilbert Schemes. The present paper is a natural extension of some of the ideas in the first part of their paper.

1. Introduction

Let R and S be rings and let $\mathsf{Mod}R$ and $\mathsf{Mod}S$ denote the category of right R-modules and right S-modules, respectively. Watts's Theorem, which was proved by Eilenberg [3] and Watts [7] at about the same time, is the following:

Theorem 1.1. *Suppose $F : \mathsf{Mod}R \to \mathsf{Mod}S$ is a right exact functor commuting with direct limits. Then $F \cong - \otimes_R B$ where B is an R-S-bimodule.*

Let $\mathsf{B}(\mathsf{Mod}R, \mathsf{Mod}S)$ denote the full subcategory of the category of functors from $\mathsf{Mod}R$ to $\mathsf{Mod}S$ consisting of right exact functors commuting with direct limits. The next result is a slightly more precise version of Theorem 1.1.

Theorem 1.2. *The functor $\Psi : \mathsf{Mod}(R^{\mathrm{op}} \otimes_{\mathbb{Z}} S) \to \mathsf{B}(\mathsf{Mod}R, \mathsf{Mod}S)$ induced by the assignment $B \mapsto - \otimes_R B$ is an equivalence of categories.*

Theorem 1.1 is then just the fact that the functor Ψ is essentially surjective.

The main result of this paper (Theorem 3.1) is that if $\mathsf{Mod}S$ is replaced by an arbitrary cocomplete[1] category A, then a version of Theorem 1.2 still holds. One of the obvious hurdles in proving such a theorem is to have a sensible notion of tensor product in this context. We use the tensor product functor that was defined in [6, Thm. 3.7.1] and investigated in detail in [1] (see Section 2.4).

In Proposition 4.2, we specialize our main result to the case that A is the category of quasi-coherent sheaves on a scheme Y. This version of the main result is used

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[1]An additive category is cocomplete if it has arbitrary direct sums. This is Grothendieck's condition Ab3.

extensively in [5] to prove a structure theorem for right exact functors between categories of quasi-coherent sheaves on schemes.

2. Preliminaries

Throughout this paper, k is a fixed commutative ring, R is a k-algebra, and $\gamma : k \to R$ is the homomorphism giving R its k-algebra structure.

2.1. **k-linearity.** Let A be an additive category. We say A is *k-linear* if for all objects X and Y in A, $\operatorname{Hom}_{\mathsf{A}}(X, Y)$ is a k-module and composition of morphisms is k-bilinear. Equivalently, A is k-linear if there is a ring homomorphism

$$c : k \to \operatorname{End}(\operatorname{id}_{\mathsf{A}})$$

from k to the ring of natural transformations from the identity functor to itself.

The first definition tells us that for each object $X \in \mathsf{A}$ and each $a \in k$ there is a morphism $a_X : X \to X$ such that

$$a_Y \circ f = f \circ a_X \tag{2-1}$$

for all $a \in k$ and $f \in \operatorname{Hom}_{\mathsf{A}}(X, Y)$. The second definition tells us there are natural transformations $c(a) : \operatorname{id}_{\mathsf{A}} \to \operatorname{id}_{\mathsf{A}}$ for each $a \in k$, and therefore associated morphisms $c(a)_X : X \to X$ for each $a \in k$ and $X \in \mathsf{A}$. The connection between the two definitions is that

$$c(a)_X = a_X$$

for all $a \in k$ and X in A.

The k-linear structure on $\mathsf{Mod}R$ is given by

$$a_M(m) = m.\gamma(a). \tag{2-2}$$

for all $M \in \mathsf{Mod}R$, $m \in M$, and $a \in k$.

2.2. **k-linear functors.** Let C and A be k-linear categories. A functor $F : \mathsf{C} \to \mathsf{A}$ is *k-linear* if the natural maps $\operatorname{Hom}_{\mathsf{C}}(X, Y) \to \operatorname{Hom}_{\mathsf{A}}(FX, FY)$ are k-linear for all X and Y in C. Equivalently, F is k-linear if F is additive and

$$F(a_Y) = a_{FY}$$

for all $a \in k$ and $Y \in \mathsf{Mod}R$.

We write

$$\mathsf{B}_k(\mathsf{C}, \mathsf{A})$$

for the full subcategory of the category of functors $\mathsf{C} \to \mathsf{A}$ consisting of k-linear right exact functors that commute with direct limits. We use the letter B to remind us of bimodules.

It is surely well known that an adjoint to a k-linear functor is again k-linear but we provide a proof of this for completeness.

Proposition 2.1. *Let C and A be k-linear categories. Let $G : \mathsf{A} \to \mathsf{C}$ be a functor having a left adjoint F. If G is k-linear so is F.*

Proof. Let $X \in \mathsf{C}$, and let

$$\nu : \operatorname{Hom}_{\mathsf{A}}(FX, FX) \to \operatorname{Hom}_{\mathsf{C}}(X, GFX)$$

be the adjoint isomorphism. By the functoriality of the adjoint isomorphisms the diagrams

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{A}}(FX,FX) & \xrightarrow{\ \nu\ } & \mathrm{Hom}_{\mathsf{C}}(X,GFX) \\ {\scriptstyle -\circ Ff}\downarrow & & \downarrow{\scriptstyle -\circ f} \\ \mathrm{Hom}_{\mathsf{A}}(FX,FX) & \xrightarrow[\ \nu\ ]{} & \mathrm{Hom}_{\mathsf{C}}(X,GFX) \end{array}$$

and

$$\begin{array}{ccc} \mathrm{Hom}_{\mathsf{A}}(FX,FX) & \xrightarrow{\ \nu\ } & \mathrm{Hom}_{\mathsf{C}}(X,GFX) \\ {\scriptstyle g\circ -}\downarrow & & \downarrow{\scriptstyle Gg\circ -} \\ \mathrm{Hom}_{\mathsf{A}}(FX,FX) & \xrightarrow[\ \nu\ ]{} & \mathrm{Hom}_{\mathsf{C}}(X,GFX) \end{array}$$

commute for all X in C, all $f \in \mathrm{Hom}_{\mathsf{C}}(X,X)$, and all $g \in \mathrm{Hom}_{\mathsf{A}}(FX,FX)$.

Let $\theta \in \mathrm{Hom}_{\mathsf{A}}(FX,FX)$ be an element in the top left corner of the diagrams. Let $f = a_X$ and $g = a_{FX}$. The commutativity therefore gives

$$\nu(\theta \circ F(a_X)) = \nu(\theta) \circ a_X \qquad \text{and}$$
$$\nu(a_{FX} \circ \theta) = G(a_{FX}) \circ \nu(\theta).$$

But $\nu(\theta) : X \to GFX$ is a k-linear morphism so $\nu(\theta) \circ a_X = a_{GFX} \circ \nu(\theta)$. Since G is k-linear, $G(a_{FX}) = a_{GFX}$. Hence

$$\nu(\theta \circ F(a_X)) = a_{GFX} \circ \nu(\theta) = G(a_{FX}) \circ \nu(\theta) = \nu(a_{FX} \circ \theta).$$

But ν is an isomorphism so

$$\theta \circ F(a_X) = a_{FX} \circ \theta.$$

Now take $\theta = \mathrm{id}_{FX}$ to get $F(a_X) = a_{FX}$, so showing that F is k-linear. □

2.3. **The category A_R.** For the remainder of this paper, we let A denote a k-linear cocomplete category.

A *left R-module in* A is a pair $(\mathcal{F}, \rho)$ where $\mathcal{F}$ is an object in A and $\rho : R \to \mathrm{End}_{\mathsf{A}} \mathcal{F}$ is a k-algebra homomorphism. Popescu [6, p. 108] calls $(\mathcal{F}, \rho)$ a left R-object of A. Let $(\mathcal{F}, \rho)$ and $(\mathcal{G}, \rho')$ be left R-modules in A. We define the set of *R-module maps* from $(\mathcal{F}, \rho)$ to $(\mathcal{G}, \rho')$ to be

$$\mathrm{Hom}_R(\mathcal{F}, \mathcal{G}) := \big\{\alpha \in \mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, \mathcal{G}) \mid \rho'(r) \circ \alpha = \alpha \circ \rho(r) \ \text{ for all } \ r \in R\big\}.$$

Using these R-module maps as morphisms we then obtain a category A_R, the category of left R-modules in A.

Suppose $(\mathcal{F}, \rho) \in \mathsf{A}_R$. If $\mathcal{G} \in \mathsf{A}$, then $\mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, \mathcal{G})$ becomes a right R-module through the composition map

$$\mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, \mathcal{G}) \times \mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, \mathcal{F}) \to \mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, \mathcal{G}),$$

i.e.,

$$\alpha.r := \alpha \circ \rho(r)$$

for $\alpha \in \mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, \mathcal{G})$ and $r \in R$. This allows us to view $\mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, -)$ as a functor $\mathsf{A} \to \mathsf{Mod} R$.

Let $\mu_x : R \to R$ be the right R-module homomorphism $\mu_x(r) := xr$.

Lemma 2.2. *Suppose* $F \in \mathsf{B}_k(\mathsf{Mod}R, \mathsf{A})$. *Define the ring homomorphism*

$$\rho : R \to \mathrm{End}_{\mathsf{A}} FR, \qquad \rho(x) := F(\mu_x).$$

Then $(FR, \rho) \in \mathsf{A}_R$.

Proof. To prove the lemma it suffices to show that ρ is a k-algebra homomorphism, i.e., that $(\rho \circ \gamma)(a) = a_{FR}$ for all $a \in k$. But

$$\rho(\gamma(a)) = F(\mu_{\gamma(a)}) = F(a_R) = a_{FR},$$

where the second equality is due to (2-2). Hence the result. □

2.4. **The functor** $- \otimes_R \mathcal{F}$**.** Recall the standing hypothesis that A is cocomplete.

Let $(\mathcal{F}, \rho) \in \mathsf{A}_R$. By [6, p. 108], the functor $\mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, -) : \mathsf{A} \to \mathsf{Mod}R$ has a left adjoint.[2] We fix a left adjoint and denote it by $- \otimes_R \mathcal{F}$. By [1, Proposition B3.1], the functor $- \otimes_R \mathcal{F}$ is unique up to isomorphism (of functors) such that

- $R \otimes_R \mathcal{F} \cong \mathcal{F}$, and
- $- \otimes_R \mathcal{F}$ is right exact and commutes with direct sums.

Since the functor $\mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, -)$ is k-linear for all $\mathcal{F} \in \mathsf{A}$, Proposition 2.1 implies the following:

Corollary 2.3. *If* $(\mathcal{F}, \rho) \in \mathsf{A}_R$, *then* $- \otimes_R \mathcal{F}$ *is* k*-linear.*

3. The generalization of Watts's Theorem

Theorem 3.1. *The functor*

$$\Psi : \mathsf{A}_R \to \mathsf{B}_k(\mathsf{Mod}R, \mathsf{A})$$

induced by the assignment

$$\Psi(\mathcal{F}) = - \otimes_R \mathcal{F},$$

is an equivalence of categories.

3.1. **The proof that** Ψ **is essentially surjective.**

Proposition 3.2. [3] *Let* $F \in \mathsf{B}_k(\mathsf{Mod}R, \mathsf{A})$. *Then* $F \cong - \otimes_R \mathcal{F}$ *where* $\mathcal{F} = FR$.

Proof. Let $\theta_M : M \to \mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, FM)$ be the composition

$$M \xrightarrow{\Lambda_M} \mathrm{Hom}_R(R, M) \xrightarrow{F} \mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, FM)$$

where Λ_M is the canonical isomorphism $m \to \lambda_m$ where $\lambda_m(r) := mr$ for all $r \in R$.

Let

$$\Theta_M : M \otimes_R \mathcal{F} \to FM$$

be the map that corresponds to θ_M under the adjoint isomorphism

$$\mathrm{Hom}_R(M, \mathrm{Hom}_{\mathsf{A}}(\mathcal{F}, FM)) \cong \mathrm{Hom}_{\mathsf{A}}(M \otimes_R \mathcal{F}, FM).$$

[2] It is essential that A be cocomplete for $- \otimes_R \mathcal{F}$ to exist. For example, if $R = \mathbb{Z}$ and A consists of finitely generated abelian groups and $\mathcal{F} = \mathbb{Z}$, there is no adjoint. But the hypothesis of cocompleteness is absent from [6, p.108] and parts of [1].

[3] After we finished writing this paper we learned that a version of this result had already been proved by Brzezinski and Wisbauer [2, 39.3, p.410] under the hypothesis that the objects of A are abelian groups.

We will show that the Θ_Ms define a natural transformation, i.e., if $f : M \to N$ is a homomorphism of right R-modules, then the diagram

$$\text{(3-1)} \qquad \begin{array}{ccc} M \otimes_R \mathcal{F} & \xrightarrow{f\otimes\mathcal{F}} & N \otimes_R \mathcal{F} \\ \downarrow{\scriptstyle \Theta_M} & & \downarrow{\scriptstyle \Theta_N} \\ FM & \xrightarrow[Ff]{} & FN \end{array}$$

commutes. Define $\eta : \operatorname{Hom}_{\mathsf{A}}(\mathcal{F}, FM) \to \operatorname{Hom}_{\mathsf{A}}(\mathcal{F}, FN)$ by $\eta(g) := Ff \circ g$. The left and right squares in the diagram

$$\begin{array}{ccccc} M & \xrightarrow{\Lambda_M} & \operatorname{Hom}_R(R, M) & \xrightarrow{F} & \operatorname{Hom}_{\mathsf{A}}(\mathcal{F}, FM) \\ \downarrow{\scriptstyle f} & & \downarrow & & \downarrow{\scriptstyle \eta} \\ N & \xrightarrow[\Lambda_N]{} & \operatorname{Hom}_R(R, N) & \xrightarrow[F]{} & \operatorname{Hom}_{\mathsf{A}}(\mathcal{F}, FN) \end{array}$$

commute, so $\eta \circ \theta_M = \theta_N \circ f$.

We now consider the diagram

$$\begin{array}{ccc} \operatorname{Hom}(M, \operatorname{Hom}(\mathcal{F}, FM)) & \xrightarrow{\sim} & \operatorname{Hom}(M \otimes \mathcal{F}, FM) \\ \downarrow & & \downarrow \\ \operatorname{Hom}(M, \operatorname{Hom}(\mathcal{F}, FN)) & \xrightarrow{\sim} & \operatorname{Hom}(M \otimes \mathcal{F}, FN) \\ \uparrow & & \uparrow \\ \operatorname{Hom}(N, \operatorname{Hom}(\mathcal{F}, FN)) & \xrightarrow{\sim} & \operatorname{Hom}(N \otimes \mathcal{F}, FN), \end{array}$$

whose verticals are induced by f and whose horizontals are the adjoint isomorphism. The top and bottom rectangles of this diagram commute by the functoriality of the adjoint isomorphisms. The maps θ_M and θ_N belong to the top and bottom Hom-sets of the left-hand column and their images in $\operatorname{Hom}(M, \operatorname{Hom}(\mathcal{F}, FN))$ are the same because $\eta \circ \theta_M = \theta_N \circ f$. It follows that the images of Θ_M and Θ_N in $\operatorname{Hom}(M \otimes \mathcal{F}, FN)$ are the same. In other words,

$$Ff \circ \Theta_M = \Theta_N \circ (f \otimes \mathcal{F})$$

which proves that (3-1) commutes and hence that the Θ_Ms define a natural transformation

$$\Theta : - \otimes_R \mathcal{F} \to F.$$

Because $(F \circ \Lambda_R)(x) = F(\mu_x) = \rho(x)$, $\theta_R : R \to \operatorname{Hom}_{\mathsf{A}}(\mathcal{F}, \mathcal{F})$ is the map giving $\mathcal{F}$ its R-module structure, so the corresponding map $\Theta_R : R \otimes_R \mathcal{F} \to \mathcal{F}$ is an isomorphism. Since the functors $- \otimes_R \mathcal{F}$ and F commute with direct sums, Θ_M is an isomorphism for all free R-modules M. Since $- \otimes_R \mathcal{F}$ and F are right exact it follows that Θ_M is an isomorphism whenever M is the cokernel of a map between free R-modules. But every R-module is of that form so Θ_M is an isomorphism for all M. Hence Θ is an isomorphism of functors.[4] □

Proposition 3.2 says that the functor Ψ in Theorem 3.1 is essentially surjective.

[4] The argument in the last part of the proof is a result of B. Mitchell. See [2, 39.1, p.409] for more details.

3.2. **The proof that Ψ is fully faithful.** Let $(\mathcal{F},\rho) \in \mathsf{A}_R$ and let $\mathcal{N} \in \mathsf{A}$. The composition

$$\operatorname{Hom}_{\mathsf{A}}(R \otimes_R \mathcal{F}, \mathcal{N}) \xrightarrow{\sim} \operatorname{Hom}_R(R, \operatorname{Hom}_{\mathsf{A}}(\mathcal{F},\mathcal{N})) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{A}}(\mathcal{F},\mathcal{N}), \tag{3-2}$$

where the first map is the adjoint isomorphism and the second is the canonical isomorphism $\psi \mapsto \psi(1)$, induces an isomorphism of functors $\operatorname{Hom}_{\mathsf{A}}(R \otimes_R \mathcal{F}, -) \to \operatorname{Hom}_{\mathsf{A}}(\mathcal{F}, -)$ which, by the Yoneda Lemma, corresponds to a unique isomorphism

$$\theta_{\mathcal{F}} : \mathcal{F} \xrightarrow{\sim} R \otimes_R \mathcal{F}.$$

The next result is a slightly sharper form of [1, Prop. B3.1(a)].

Proposition 3.3. *The diagram*

$$\begin{array}{ccc} R \otimes_R \mathcal{F} & \xrightarrow{R\otimes\phi} & R \otimes_R \mathcal{G} \\ {\scriptstyle \theta_{\mathcal{F}}}\uparrow & & \uparrow{\scriptstyle \theta_{\mathcal{G}}} \\ \mathcal{F} & \xrightarrow[\phi]{} & \mathcal{G} \end{array} \tag{3-3}$$

commutes for all $\mathcal{F},\mathcal{G} \in \mathsf{A}_R$ and all $\phi \in \operatorname{Hom}_R(\mathcal{F},\mathcal{G})$. Therefore, the maps $\theta_{\mathcal{F}}$ provide an isomorphism

$$\theta : \mathrm{id}_{\mathsf{A}_R} \longrightarrow (R \otimes_R -)$$

of functors.

Proof. By the Yoneda lemma, the commutivity of (3-3) is equivalent to the condition that for all $\mathcal{N} \in \mathsf{A}$ the outer rectangle in the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathsf{A}}(R \otimes_R \mathcal{F}, \mathcal{N}) & \xleftarrow{-\circ(R\otimes\phi)} & \operatorname{Hom}_{\mathsf{A}}(R \otimes_R \mathcal{G}, \mathcal{N}) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \operatorname{Hom}_R(R, \operatorname{Hom}_{\mathsf{A}}(\mathcal{F},\mathcal{N})) & \xleftarrow{\Gamma} & \operatorname{Hom}_R(R, \operatorname{Hom}_{\mathsf{A}}(\mathcal{G},\mathcal{N})) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ \operatorname{Hom}_{\mathsf{A}}(\mathcal{F},\mathcal{N}) & \xleftarrow[-\circ\phi]{} & \operatorname{Hom}_{\mathsf{A}}(\mathcal{G},\mathcal{N}) \end{array} \tag{3-4}$$

commutes, where the vertical arrows are the factorizations in (3-2) that are used to define $\theta_{\mathcal{F}}$ and $\theta_{\mathcal{G}}$, and

$$\Gamma(\psi)(x) := \psi(x) \circ \phi$$

for all $x \in R$ and $\psi \in \operatorname{Hom}_R(R, \operatorname{Hom}_{\mathsf{A}}(\mathcal{G},\mathcal{N}))$.

The uppermost square of (3-4) commutes by functoriality of the adjoint isomorphism. Going clockwise around the lower square, the image in $\operatorname{Hom}_{\mathsf{A}}(\mathcal{F},\mathcal{N})$ of $\psi \in \operatorname{Hom}_R(R, \operatorname{Hom}_{\mathsf{A}}(\mathcal{G},\mathcal{N}))$ is $\psi(1) \circ \phi$. Going counter-clockwise around the lower square, the image of ψ in $\operatorname{Hom}_{\mathsf{A}}(\mathcal{F},\mathcal{N})$ is $\Gamma(\psi)(1) = \psi(1) \circ \phi$. Hence the lower square commutes.

It follows that the outer rectangle commutes. □

Lemma 3.4. *Let C be a cocomplete abelian category and let $F, G : \mathsf{Mod}R \to \mathsf{C}$ be right exact functors that commute with direct sums. Let $\tau, \tau' : F \to G$ be natural transformations. If $\tau_R = \tau'_R$, then $\tau = \tau'$.*

Proof. Let M_i, $i \in I$, be a collection of right R-modules. Then there is a natural map

$$\bigoplus_{i\in I} FM_i \to F\Big(\bigoplus_{i\in I} M_i\Big)$$

and the fact that F commutes with direct sums says that this map is an isomorphism. By the universal property of colimits, there is a commutative diagram

$$\begin{array}{ccc} \bigoplus_{i\in I} FM_i & \longrightarrow & F\Big(\bigoplus_{i\in I} M_i\Big) \\ {\scriptstyle \oplus\tau_{M_i}}\downarrow & & \downarrow{\scriptstyle \tau_{\oplus M_i}} \\ \bigoplus_{i\in I} GM_i & \longrightarrow & G\Big(\bigoplus_{i\in I} M_i\Big). \end{array}$$

Since the horizontal maps are isomorphisms, if $\tau_{M_i} = \tau'_{M_i}$ for all i, then

$$\tau_{\oplus M_i} = \tau'_{\oplus M_i}.$$

In particular, it follows that $\tau_P = \tau'_P$ for all free R-modules P.

Let M be a right R-module and let $P \to Q \to M \to 0$ be an exact sequence in which P and Q are free R-modules. Then there is a commutative diagram

$$\begin{array}{ccccccc} FP & \longrightarrow & FQ & \longrightarrow & FM & \longrightarrow & 0 \\ {\scriptstyle \tau_P}\downarrow & & {\scriptstyle \tau_Q}\downarrow & & & & \\ GP & \longrightarrow & GQ & \longrightarrow & GM & \longrightarrow & 0, \end{array}$$

and a unique map $FM \to GM$ making the diagram commute, namely τ_M. Since $\tau_P = \tau'_P$ and $\tau_Q = \tau'_Q$, it follows that $\tau_M = \tau'_M$. □

Now we prove that Ψ is fully faithful. Let $\mathcal{F}$ and $\mathcal{G}$ be objects in A_R and let $\phi \in \operatorname{Hom}_R(\mathcal{F}, \mathcal{G})$. By Proposition 3.3, the diagram

(3-5)
$$\begin{array}{ccc} R \otimes_R \mathcal{F} & \xrightarrow{R\otimes\phi} & R \otimes_R \mathcal{G} \\ {\scriptstyle \theta_{\mathcal{F}}}\uparrow & & \uparrow{\scriptstyle \theta_{\mathcal{G}}} \\ \mathcal{F} & \xrightarrow[\phi]{} & \mathcal{G} \end{array}$$

commutes. It follows from this that $R \otimes \phi$ is non-zero if ϕ is non-zero. Hence Ψ is faithful.

To complete the proof of Theorem 3.1, it remains to show that Ψ is full. To that end, let

$$\tau : - \otimes_R \mathcal{F} \to - \otimes_R \mathcal{G}$$

be a natural transformation. We must show there is a homomorphism $\phi \in \operatorname{Hom}_R(\mathcal{F}, \mathcal{G})$ such that $\tau_M = M \otimes \phi$ for all $M \in \mathsf{Mod}R$.

Define

$$\phi := \theta_{\mathcal{G}}^{-1} \circ \tau_R \circ \theta_{\mathcal{F}}.$$

It follows from the commutativity of (3-5) that $R \otimes \phi = \tau_R$. By Lemma 3.4, it follows that $M \otimes \phi = \tau_M$ for all $M \in \mathsf{Mod}R$. In other words, $\Psi(\phi) = \tau$.

4. An application

Throughout this section, let X denote a k-scheme. If $X = \operatorname{Spec} R$, we let

$$\widetilde{(-)} : \mathsf{Mod}R \to \mathsf{Qcoh}X$$

be the quasi-inverse to the global sections functor defined in [4, II, Definition, p. 110].

Example 4.1. Let $f : Y \to X$ be a morphism from an arbitrary scheme to an affine scheme $X = \operatorname{Spec} R$. Then $f^* \circ \widetilde{(-)} : \mathsf{Mod}R \to \mathsf{Qcoh}Y$ is a right exact functor commuting with direct sums. Proposition 3.2 says that $f^* \circ \widetilde{(-)} \cong - \otimes_R \mathcal{O}_Y$ where $\mathcal{O}_Y$ is made into an R-module via the ring homomorphism

$$R \to \operatorname{Hom}_R(R, R) \to \operatorname{Hom}_Y(f^*\mathcal{O}_X, f^*\mathcal{O}_X) \to \operatorname{Hom}_Y(\mathcal{O}_Y, \mathcal{O}_Y)$$

where the first map sends $r \in R$ to multiplication by r, the second map is induced by $f^* \circ \widetilde{(-)}$ and the third isomorphism is induced by the natural isomorphism $f^*\mathcal{O}_X \cong O_Y$.

The motivation for this paper lies in the paper [5], in which k-schemes X and Y and k-linear functors $F : \mathsf{Qcoh}X \to \mathsf{Qcoh}Y$ that are right exact and commute with direct sums are considered. One source of such functors is the following. Let $\mathcal{F}$ be a quasi-coherent sheaf on $X \times_k Y$, and define

$$- \otimes_{\mathcal{O}_X} \mathcal{F} := \operatorname{pr}_{2*}(\operatorname{pr}_1^*(-) \otimes_{\mathcal{O}_{X \times_k Y}} \mathcal{F}) \qquad (4\text{-}1)$$

where $\operatorname{pr}_i : X \times_k Y \to X, Y$, $i = 1, 2$, are the obvious projections. A functor of the form $- \otimes_{\mathcal{O}_X} \mathcal{F}$ is not always an object of $\mathsf{B}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$. This happens, for example, if $Y = \operatorname{Spec} k$, $X = \mathbb{P}^1_k$ and $F = - \otimes_{\mathcal{O}_X} \mathcal{O}_{X \times Y} \cong \Gamma(X, -)$.

On the other hand, an object of $\mathsf{B}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y)$ is not always isomorphic to one of the form $- \otimes_{\mathcal{O}_X} \mathcal{F}$. This happens, for example, if $Y = \operatorname{Spec} k$, $X = \mathbb{P}^1_k$ and $F = H^1(X, -)$ [5, Proposition 5.4].

The question motivating [5] is whether F is isomorphic to a functor of the form $- \otimes_{\mathcal{O}_X} \mathcal{F}$. It follows from Theorem 3.1 that this is always the case if X is affine, as we now show.

Proposition 4.2. *Let R be a k-algebra and Y a k-scheme. Write $X := \operatorname{Spec} R$. Then the inclusion functor*

$$\mathsf{Qcoh}(X \times_k Y) \to \mathsf{B}_k(\mathsf{Qcoh}X, \mathsf{Qcoh}Y), \qquad \mathcal{F} \mapsto - \otimes_{\mathcal{O}_X} \mathcal{F},$$

is an equivalence of categories.

Proof. By [4, II, exercise 5.17e], the functor

$$\operatorname{pr}_{2*} : \mathsf{Qcoh}(X \times_k Y) \to \mathsf{Qcoh}(\operatorname{pr}_{2*} \mathcal{O}_{X \times_k Y})$$

is an equivalence, where $\mathsf{Qcoh}(\operatorname{pr}_{2*} \mathcal{O}_{X \times_k Y})$ denotes the category of quasi-coherent $\mathcal{O}_Y$-modules with $\operatorname{pr}_{2*} \mathcal{O}_{X \times_k Y}$-module structure. Furthermore, it is straightforward to check that the functor

$$\mathsf{Qcoh}(\operatorname{pr}_{2*} \mathcal{O}_{X \times_k Y}) \to (\mathsf{Qcoh}Y)_R$$

induced by the assignment $\mathcal{E} \mapsto (\mathcal{E}, \rho)$, where $\rho : R \to \operatorname{Hom}_Y(\mathcal{E}, \mathcal{E})$ is defined through the $\operatorname{pr}_{2*} \mathcal{O}_{X \times_k Y}$-structure of $\mathcal{E}$, is an equivalence. By Theorem 3.1, the functor

$$(\mathsf{Qcoh}Y)_R \to \mathsf{B}_k(\mathsf{Mod}R, \mathsf{Qcoh}Y)$$

induced by the assignment $(\mathcal{E}, \rho) \mapsto -\otimes_R \mathcal{E}$ is an equivalence. Therefore, the functor

$$\mathsf{Qcoh}(X \times_k Y) \to \mathsf{B}_k(\mathsf{Mod}R, \mathsf{Qcoh}Y)$$

induced by the assignment $\mathcal{F} \mapsto - \otimes_R \mathrm{pr}_{2*} \mathcal{F}$ is an equivalence. By the uniqueness properties of the functor $- \otimes_R \mathcal{E}$ described in Section 2.4, we have an isomorphism of functors

$$- \otimes_R \mathrm{pr}_{2*} \mathcal{F} \xrightarrow{\sim} \widetilde{(-)} \otimes_{\mathcal{O}_X} \mathcal{F}$$

in $\mathsf{B}_k(R, \mathsf{Qcoh}Y)$. It follows that the functor

$$\mathsf{Qcoh}(X \times_k Y) \to \mathsf{B}_k(\mathsf{Mod}R, \mathsf{Qcoh}Y)$$

induced by the assignment $\mathcal{F} \mapsto \widetilde{(-)} \otimes_{\mathcal{O}_X} \mathcal{F}$ is an equivalence. The claim follows easily from this. □

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