Species and noncommutative projective lines over non-algebraic bimodules

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Heuristic

Points of $\mathbb{P}(V) \to$ Indecomposable Λ-modules

Beilinson's Theorem

Theorem (Beilinson 1978)

The functor $RHom(\mathcal{O} \oplus \mathcal{O}(1), -)$ gives an equivalence

$$D^b(\text{coh}\mathbb{P}(V)) \to D^b(\text{mod}\Lambda).$$

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Remark

- indecomp. vector bundles $\longleftrightarrow$ modules of dimension type (a, b), $|a - b| = 1$.
- indecomp. torsion modules $\longleftrightarrow$ modules of dimension type (n, n)

Bimodule Species

Definition

K_0, K_1 fields (with char $\neq 2$), $V = K_0 - K_1$-bimodule with left-right dimension two. The **bimodule species** corresponding to V is the algebra

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Question

What are the indecomposable $\Lambda(V)$-modules?

Definition

V **algebraic** if there is subfield k of K_0 and K_1 which acts centrally on V and such that K_i/k is finite.

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- $\exists!$ indecomposable of dim. type (a, b) for each pair with $|a - b| = 1$.
- All others have type (n, n). They form category equivalent to

$$\mathsf{T} \times \mathsf{F}$$

  T uniserial w/ one simple object, $\mathsf{F} =$ f.l. modules over $K_0[x; \sigma, \delta]$.

Our Main Idea

We prove there is a correspondence

$$\text{Indecomposables in } \mathrm{coh}\mathbb{P}^{nc}(V) \longleftrightarrow \text{Indecomposables in } \mathrm{mod}\Lambda(V)$$

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- $\text{gr}A$ is category of f.g. graded A-modules and
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Motivation:

Theorem (Serre 1955)

If k is a field, A is a f.g. commutative k-algebra generated in degree one and X is the associated scheme, then

$$\text{proj}A \equiv \text{coh}X$$

What is $\mathbb{P}^{nc}(V)$? Part II

$\mathbb{S}(W)$

Recall that for W a vector space over a field K,

$$\mathbb{S}(W) := \frac{K \oplus W \oplus W^{\otimes 2} \oplus \cdots}{\langle x \otimes y - y \otimes x \rangle}$$

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Want noncommutative ring $\mathbb{S}^{nc}(V)$ depending only on $K_0 - K_1$-bimodule V so we can define (after M. Van den Bergh)

$$\text{coh}\mathbb{P}^{nc}(V) := \text{proj}\mathbb{S}^{nc}(V).$$

Heuristic

An algebra A is a $\mathbb{Z}$-**algebra** if $A = \oplus_{(i,j)\in\mathbb{Z}^2} A_{ij}$ and $A_{ij}A_{jk} \subset A_{ik}$.

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- mult. induced by $\otimes_K$.

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Consequence

Indecomposables in coh$\mathbb{P}^{nc}(V) \longleftrightarrow$ Indecomposables in mod$\Lambda(V)$

- $\mathbb{S}^{nc}(V)$ has normal "element" g
- $g = 0$ corresponds to commutative point in $\mathbb{P}^{nc}(V)$

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- T is uniserial with one simple object and
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Heuristic

T corresponds to sheaves supported on $g = 0$ while F corresponds to sheaves supported on the (affine open) complement $g \neq 0$.

Thank you for your attention!