TWO-SIDED VECTOR SPACES

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Abstract. We study the structure of two-sided vector spaces over a perfect field K. In particular, we give a complete characterization of isomorphism classes of simple two-sided vector spaces which are left finite-dimensional. Using this description, we compute the Quillen K-theory of the category of left finite-dimensional, two-sided vector spaces over K. We also consider the closely related problem of describing homomorphisms $\phi : K \to M_n(K)$.

1. Introduction

Given the central role that vector spaces play in mathematics, it is natural to study two-sided vector spaces; that is, abelian groups V equipped with both a left and right action by a field K, subject to the associativity condition $(xv)y = x(vy)$ for $x, y \in K$ and $v \in V$. When the left and right actions of K on V agree, then V is nothing more than an ordinary K-vector space. In this case, V decomposes into a direct sum of irreducible subspaces, and every irreducible subspace is 1-dimensional (and hence isomorphic to K as a vector space over K). When the left and right actions of K and V differ, then the structure of V can be much more complicated. For example, V does not generally decompose into irreducible subspaces. Furthermore, the distinct irreducible subspaces of V may not be 1-dimensional or isomorphic to each other.

Apart from being intrinsically interesting, two-sided vector spaces play an important role in noncommutative algebraic geometry. In particular, two-sided vector spaces are noncommutative analogues of vector bundles over $\operatorname{Spec} K$. Noncommutative analogues of vector bundles were defined and used by Van den Bergh [9] to construct noncommutative $\mathbb{P}^1$-bundles over commutative schemes.

The purpose of this paper is to study the structure of two-sided vector spaces over K when K is a perfect field. In particular, we classify irreducible two-sided vector spaces which are finite-dimensional as ordinary K-vector spaces. We then use our classification to determine the algebraic K-theory of the category of all such two-sided vector spaces. We also give canonical representations for certain two-sided vector spaces, generalizing [5, Theorem 1.3].

The structure theory of two-sided vector spaces has important applications to noncommutative algebraic geometry via the theory of noncommutative vector bundles. Let S and X be commutative schemes and suppose X is an S-scheme of

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finite type. By an "S-central noncommutative vector bundle over X" we mean an $\mathcal{O}_S$-central, coherent sheaf X-bimodule which is locally free on the right and left [9, Definition 2.3, p. 440]. When $S = \operatorname{Spec} k$ and $X = \operatorname{Spec} K$, a sheaf X-bimodule which is locally free of finite rank on each side is nothing more than a two-sided K-vector space V, finite-dimensional on each side, where the left and right actions of K on V may differ.

When X is an integral scheme, any noncommutative vector bundle $\mathcal{E}$ over X localizes to a noncommutative vector bundle $\mathcal{E}_\eta$ over the generic point η of X. If $\mathcal{O}_X$ acts centrally on $\mathcal{E}$, then $\mathcal{E}_\eta$ is completely characterized by its dimension over the field of fractions, $k(X)$, of X. In this case, the rank of $\mathcal{E}$ is defined as $\dim_{k(X)} \mathcal{E}_\eta$. Since localization is exact, localization induces a map $K_0(X) \to K_0(\operatorname{Spec} k(X))$, and the rank of $\mathcal{E}$ can also be defined as the image of the class of $\mathcal{E}$ via this map.

Now suppose X is of finite type over $\operatorname{Spec} k$. If $\mathcal{O}_X$ does not act centrally on $\mathcal{E}$, then $\mathcal{E}_\eta$ will be a two-sided vector space over $k(X)$ whose left and right actions differ. In this case, $\mathcal{E}_\eta$ is not completely characterized by its left and right dimension. However, localization induces a map $K_0^B(X) \to K_0^B(\operatorname{Spec} k(X))$ where $K_0^B(X)$ denotes the Quillen K-theory of the category of k-central noncommutative vector bundles over X and $K_0^B(\operatorname{Spec} k(X))$ is defined similarly. It is thus reasonable to define the rank of $\mathcal{E}$ as the image of the class of $\mathcal{E}$ via this map. If this notion of rank is to be useful we must be able to compute the group $K_0^B(\operatorname{Spec} k(X))$.

In addition, one can often construct a noncommutative symmetric algebra $\mathcal{A}$ from a noncommutative vector bundle $\mathcal{E}$ [5, Section 2], [8, Section 5.1]. While $\mathcal{A}$ is not generally a sheaf of algebras over X, its localization at the generic point η of X, $\mathcal{A}_\eta$, is an algebra. The birational class of the projective bundle associated to $\mathcal{A}$ is determined by the degree zero component of the skew field of fractions of $\mathcal{A}_\eta$. Since $\mathcal{A}_\eta$ is generated by $\mathcal{E}_\eta$, we see that the birational class of a noncommutative projectivization is governed by a noncommutative vector bundle over $\operatorname{Spec} K(X)$.

We now summarize the contents of the paper. In Section 2 we describe some general properties of two-sided vector spaces that we will use in the sequel. In Section 3 we study simple objects in $\mathsf{Vect}(K)$, the category of two-sided K-vector spaces which are left finite-dimensional. In particular, we parameterize isomorphism classes of simple two-sided vector spaces by orbits of embeddings $\lambda : K \to \bar{K}$ under the action of left-composition by elements of $\operatorname{Aut}(\bar{K}/K)$ (Theorem 3.2). In Section 4, we use results from Section 3 to explicitly describe the Quillen K-groups of $\mathsf{Vect}(K)$, denoted $K_i^B(K)$ (Theorem 4.1), and give a procedure for calculating the ring structure on $K_0^B(K)$.

Finally in Section 5, we study matrix representations of two-sided vector spaces, i.e. homomorphisms $\phi : K \to M_n(K)$. Specifically, we consider the problem of finding a $P \in GL_n(K)$ such that the homomorphism $P\phi P^{-1}$ has a particularly nice form. We prove that if every matrix in $\operatorname{im} \phi$ has all of its eigenvalues in K, then the triangularized form of ϕ can be described in terms of higher derivations on K (Theorem 5.4). We also develop sufficient conditions on a matrix A to ensure the existence of an upper triangular matrix $P \in GL_n(K)$ with PAP^{-1} in Jordan canonical form (Theorem 5.8). Combining these results, we give sufficient conditions that enable us to describe the off diagonal blocks of $P\phi P^{-1}$ (Corollary 5.10).

Throughout the paper, we provide examples of our results. We reproduce and extend the third case of [5, Theorem 1.3] by describing the structure of 2 and 3-dimensional simple two-sided vector spaces when they exist. When $p \geq 3$ is prime and $K = \mathbb{Q}(\sqrt[p]{2})$, we describe the isomorphism classes of $\mathbb{Q}$-central two-sided K-vector spaces. There are only two, with dimensions 1 and $p-1$. We then describe the ring $K_0^B(K)$ via generators and relations. Finally, we provide an example in Section 5 to show that there exists a field K, a homomorphism $\phi : K \to M_3(K)$, and an element $y \in K$ such that there is no $P \in GL_3(K)$ with $P\phi P^{-1}$ upper triangular and $P\phi(y)P^{-1}$ in Jordan canonical form (Example 5.5).

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2. Preliminaries

As we mentioned above, K will always denote a perfect field of arbitrary characteristic and $\bar{K}$ will be a fixed algebraic closure of K. By a *two-sided vector space* we mean a K-bimodule V where the left and right actions of K on V do not necessarily coincide. Except when explicitly stated to the contrary, we shall only consider those two-sided vector spaces whose left dimension is finite, and we use the phrases "two-sided vector space" and "bimodule" interchangeably.

Since we shall only consider bimodules V with ${}_KV$ and V_K both unital, it is easy to see that the prime subfield of K must act centrally on any two-sided vector space. We shall fix a base field $k \subset K$ and consider only those bimodules V which are centralized by k. Note that we do not assume that K/k is algebraic in general. While all of the notions that we introduce in this paper will depend on the centralizing subfield k, it turns out that k itself will usually not play an important role in any of our results. In particular we will omit k from our notation.

Given a K-bimodule V and a set of vectors $\{v_i : i \in I\}$, we shall always write $\operatorname{span}\{v_i\}$ to stand for the *left* span of the v_i. In general, $\operatorname{span}\{v_i\}$ will not be a sub-bimodule of V.

If V is a two-sided vector space, then right multiplication by $x \in K$ defines an endomorphism $\phi(x)$ of ${}_KV$, and the right action of K on V is via the k-algebra homomorphism $\phi : K \to \operatorname{End}({}_KV)$. This observation motivates the following definition.

Definition 2.1. Let $\phi : K \to M_n(K)$ be a nonzero homomorphism. Then we denote by ${}_1K_\phi^n$ the two-sided vector space of left dimension n, where the left action is the usual one and the right action is via ϕ; that is,

$$x \cdot (v_1, \dots, v_n) = (xv_1, \dots, xv_n), \quad (v_1, \dots, v_n) \cdot x = (v_1, \dots, v_n)\phi(x). \tag{1}$$

We shall always write scalars as acting to the left of elements of ${}_1K_\phi^n$ and matrices acting to the right; thus, elements of K^n are written as row vectors and if $v \in K^n$ is an eigenvector for $\phi(x)$ with eigenvalue λ, we write $v\phi(x) = \lambda v$.

It is easy to see that, if V is a two-sided vector space and $[K : k] < \infty$, then $\dim {}_KV$ is finite if and only if $\dim V_K$ is finite, and in this case the two dimensions must be equal. Thus, when $[K : k] < \infty$, we may drop subscripts and simply write $\dim V$ for this common dimension. If $[K : k]$ is infinite, it is no longer true that the finiteness of $\dim {}_K V$ implies the finiteness of $\dim V_K$, as the following example shows.

Example 2.2. Let $K = k(x_1, x_2, \ldots)$, let $\phi : K \to K$ be the homomorphism defined by $\phi(x_i) = x_{i+1}$ and let $V = {}_1K_\phi$. Then the dimension of ${}_KV$ is 1, while the dimension of V_K is infinite.

We denote the category of left finite-dimensional two-sided vector spaces by $\mathsf{Vect}(K)$. Clearly $\mathsf{Vect}(K)$ is a finite-length category. If we write $K^e = K \otimes_k K$ for the enveloping algebra of K, then there is a category equivalence between (not necessarily finite-dimensional) K-bimodules and (say) left K^e-modules. Under this equivalence, $\mathsf{Vect}(K)$ can be identified as a full subcategory of the category of finite-length K^e-modules. If $[K : k]$ is finite, then $\mathsf{Vect}(K) = K^e\text{-}\operatorname{mod}$, the category of noetherian left K^e-modules. When K/k is infinite, this need no longer hold: if we define $V = {}_\phi K_1$ in the obvious way for the map ϕ in Example 2.2, then V is clearly simple in $K^e\text{-}\operatorname{Mod}$ but is not in $\mathsf{Vect}(K)$.

If $V \in \mathsf{Vect}(K)$ with left dimension equal to n, then choosing a left basis for V shows that $V \cong {}_1K^n_\phi$ for some homomorphism $\phi : K \to M_n(K)$; we shall say that ϕ *represents* V in this case.

If L is an extension field of K, then of course any matrix over K can be viewed as a matrix over L, and a function $\phi : K \to M_n(K)$ can be viewed as having its image in $M_n(L)$. If $A, B \in M_n(K)$, then we write $A \sim_L B$ if A and B are similar in $M_n(L)$; that is, if $B = PAP^{-1}$ for some $P \in GL_n(L)$. Similarly, if $\phi : K \to M_n(K)$ and $\psi : K \to M_n(K)$ are functions, we write $\phi \sim_L \psi$ if $\phi(x) = P\psi(x)P^{-1}$ for some $P \in GL_n(L)$. In either case, if P actually lives in $M_n(K)$, then we simply write $\sim$ for $\sim_K$.

The following well known result follows readily from the fact that a homomorphism $\phi : K \to M_n(K)$ restricts to a representation of the group K^* of units of K.

Lemma 2.3. *Let L be an extension field of K. $L \otimes_K {}_1K^n_\phi \cong L \otimes_K {}_1K^n_\psi$ as $L \otimes_K K^e$-modules if and only if $\phi \sim_L \psi$.*

The next result is a special case of the Noether-Deuring Theorem [1, Exercise 6, p. 139].

Lemma 2.4. *Let L be an extension field of K, and let $A, B \in M_n(K)$. If $A \sim_L B$, then $A \sim B$. Similarly, if $\phi : K \to M_n(K)$ and $\psi : K \to M_n(K)$ are functions with $\phi \sim_L \psi$, then $\phi \sim \psi$.*

3. Simple two-sided vector spaces

The main result of this section is a determination of all of the isomorphism classes of simple two-sided vector spaces. In order to state our classification, we introduce some notation. We write $\operatorname{Emb}(K)$ for the set of k-embeddings of K into $\bar{K}$, and $G = G(K)$ for the absolute Galois group $\operatorname{Aut}(\bar{K}/K)$. (Note that $\bar{K}/K$ is Galois since K is perfect.) If L is an intermediate field, then we write $G(L)$ for $\operatorname{Aut}(\bar{K}/L)$.

Now, G acts on $\operatorname{Emb}(K)$ by left composition. Given $\lambda \in \operatorname{Emb}(K)$, we denote the orbit of λ under this action by λ^G, and we write $K(\lambda)$ for the composite field $K \vee \operatorname{im}(\lambda)$. The stabilizer G_λ of λ under this action is easy to calculate: $\sigma\lambda = \lambda$ if and only if σ fixes $\operatorname{im}(\lambda)$; since σ fixes K as well we have that $G_\lambda = G(K(\lambda))$.

Lemma 3.1. *$[K(\lambda) : K]$ is finite if and only if $|\lambda^G|$ is finite, and in this case $|\lambda^G| = [K(\lambda) : K]$.*

Proof. By the above, the stabilizer of λ is $G(K(\lambda))$. Thus $|\lambda^G| = [G : G(K(\lambda))]$. The result now follows by basic Galois Theory. □

It turns out that we will only be interested in those embeddings λ with λ^G finite; we denote the set of finite orbits of $\mathrm{Emb}(K)$ under the action of G by $\Lambda(K)$. The following theorem gives our classification of simple bimodules.

Theorem 3.2. *There is a one-to-one correspondence between isomorphism classes of simples in $\mathsf{Vect}(K)$ and $\Lambda(K)$. Moreover, if V is a simple two-sided vector space corresponding to $\lambda^G \in \Lambda(K)$, then $\dim_K V = |\lambda^G|$ and $\mathrm{End}(V) \cong K(\lambda)$.*

To prove the first part of Theorem 3.2, we construct a map from the collection of simple bimodules to $\Lambda(K)$ and show that it gives the desired bijection. We begin in greater generality, starting with a (not necessarily simple) two-sided vector space V with $V \cong {}_1K^n_\phi$. Now, $\mathrm{im}\,\phi$ is a set of pairwise commuting matrices in $M_n(K)$; viewing $\mathrm{im}\,\phi$ as a subset of $M_n(\bar{K})$, we know that there exists a common eigenvector $v \in \bar{K}^n$ for $\mathrm{im}\,\phi$. Define a function $\lambda : K \to \bar{K}$ by letting $\lambda(x)$ be the eigenvalue of $\phi(x)$ corresponding to v; i.e. $v\phi(x) = \lambda(x)v$. It is easy to check that λ is an embedding of K into $\bar{K}$, and since ϕ is a k-algebra homomorphism we have that $\lambda \in \mathrm{Emb}(K)$.

Lemma 3.3. *If $v \in \bar{K}^n$ is a common eigenvector for $\mathrm{im}\,\phi$ with corresponding eigenvalue λ, then $\lambda \in \Lambda(K)$. Moreover, $|\lambda^G| \leq n$.*

Proof. Note first that if $\sigma \in G$, $\sigma(v)$ is also a common eigenvector of $\mathrm{im}\,\phi$, with corresponding eigenvalue $\sigma\lambda$. Indeed, we compute

$$\sigma(v)\phi(x) = \sigma(v)\sigma(\phi(x)) = \sigma(v\phi(x)) = \sigma(\lambda(x)v) = \sigma\lambda(x)\sigma(v). \tag{2}$$

Now, if $\sigma\lambda \neq \tau\lambda$, then for at least one value of $x \in K$ the vectors $\sigma(v)$ and $\tau(v)$ are eigenvectors for $\phi(x)$ with different eigenvalues; from this it follows that $\sigma(v)$ and $\tau(v)$ are linearly independent. If $\lambda^G = \{\sigma_i\lambda : i \in I\}$, then $\{\sigma_i(v) : i \in I\}$ is a linearly independent subset of $\bar{K}^n$. Thus $|\lambda^G| \leq n$ and in particular $\lambda^G \in \Lambda(K)$. □

Viewing λ as an embedding of K into $K(\lambda)$, we may without loss of generality assume that the common eigenvector v for $\mathrm{im}\,\phi$ with eigenvalue λ lives in $K(\lambda)^n$. We now fix notation which will be useful when proving Theorem 3.2. We let $m = [K(\lambda) : K] = |\lambda^G|$ and we fix a basis $\{\alpha_1, \ldots, \alpha_m\}$ for $K(\lambda)/K$. We may write

$$v = \sum_{i=1}^{m} \alpha_i v_i \tag{3}$$

with each $v_i \in K^n$ and

$$\lambda(x) = \sum_{i=1}^{m} \lambda_i(x)\alpha_i$$

where each $\lambda_i : K \to K$ is an additive function. Finally, we let β_{ijk} denote the structure constants for the basis $\{\alpha_1, \ldots, \alpha_m\}$; that is,

$$\alpha_i\alpha_j = \sum_{k=1}^{m} \beta_{ijk}\alpha_k.$$

Lemma 3.4. *In the above notation,* $\operatorname{span}\{v_1,\dots,v_m\}$ *is a two-sided subspace of* V*. In particular, if* V *is simple,* $\dim_K V = |\lambda^G|$.

Proof. We must show that $v_i\phi(x) \in \operatorname{span}\{v_1,\dots,v_m\}$ for all $x \in K$ and all i. On the one hand, $v\phi(x) = \bigl(\sum_i \alpha_i v_i\bigr)\phi(x) = \sum_i \alpha_i v_i \phi(x)$. On the other hand,

$$\begin{aligned} v\phi(x) = \lambda(x)v &= \bigl(\sum_p \lambda_p(x)\alpha_p\bigr)\bigl(\sum_q \alpha_q v_q\bigr) \\ &= \sum_{p,q} \lambda_p(x)\alpha_p\alpha_q v_q = \sum_i \alpha_i\bigl(\sum_{p,q}\beta_{pqi}\lambda_p(x)v_q\bigr). \end{aligned} \tag{4}$$

Matching up coefficients of α_i shows that $v_i\phi(x) = \sum_{p,q}\beta_{pqi}\lambda_p(x)v_q$, so that $v_i\phi(x) \in \operatorname{span}\{v_1,\dots,v_m\}$. This proves the first assertion.

If V is simple, the first part of the lemma implies $V = \operatorname{span}\{v_1,\dots,v_m\}$. Thus, $m = |\lambda^G| \geq \dim_K V$. On the other hand, $|\lambda^G| \leq \dim_K V$ by Lemma 3.3. Thus, $|\lambda^G| = \dim_K V$ when V is simple. □

Proposition 3.5. *Let* $\phi : K \to M_n(K)$ *be a homomorphism and let* $\lambda : K \to \overline{K}$ *be the eigenvalue of a common eigenvector of* $\operatorname{im}\phi \subset M_n(\overline{K})$*. The map*

$$\Phi : \{\textit{Isomorphism classes of simples in } \mathsf{Vect}(K)\} \to \Lambda(K)$$

defined by $\Phi([{}_1K^n_\phi]) = \lambda^G$ *is a bijection.*

Proof. Part 1. We show Φ is an injection.

Part 1, Step 1. We show Φ *is well defined.* Let V be a simple object in $\mathsf{Vect}(K)$, and suppose $V \cong {}_1K^n_\phi$. By Lemma 3.4, $|\lambda^G| = n$. Let us write out the elements of λ^G as $\{\lambda, \sigma_2\lambda, \dots, \sigma_n\lambda\}$. Then taking $\{v, \sigma_2(v), \dots, \sigma_n(v)\}$ as a basis for $\bar{K}^n$, we see that there exists $Q \in GL_n(\bar{K})$ such that

$$Q\phi(x)Q^{-1} = \operatorname{diag}(\lambda(x), \sigma_2\lambda(x), \dots, \sigma_n\lambda(x)) \tag{5}$$

for all $x \in K$. In particular, if $\mu : K \to \bar{K}$ is the eigenvalue for $\phi(x)$ corresponding to some common eigenvector w of $\operatorname{im}\phi$, then we must have $\mu = \sigma_i\lambda$ for some i; that is, $\mu^G = \lambda^G$.

If we choose a different isomorphism $V \cong {}_1K^n_\psi$, then $\phi \sim \psi$; say $\phi \cong P\psi P^{-1}$ for some $P \in GL_n(K)$. If v is a common eigenvector for $\operatorname{im}\phi$ with corresponding eigenvalue λ, then an easy computation shows that vP is a common eigenvector for $\operatorname{im}(\psi)$ with corresponding eigenvalue λ.

Part 1, Step 2. We show Φ *is an injection.* If K is finite, then every embedding of K into $\bar{K}$ is in fact an automorphism of K. Hence every simple in $\mathsf{Vect}(K)$ is isomorphic to ${}_1K_\phi$ for some $\phi \in \operatorname{Aut}(K)$, and the above correspondence just sends ${}_1K_\phi$ to ϕ. Thus the claim follows when K is finite.

Now suppose that K is infinite, $\Phi([V]) = \lambda^G = \Phi([W])$ and $|\lambda^G| = n$. Write $V \cong {}_1K^n_\phi$ and $W \cong {}_1K^n_\psi$. As in equation (5), there are invertible matrices $P, Q \in M_n(\bar{K})$ such that

$$P\phi(x)P^{-1} = Q\psi(x)Q^{-1} = \operatorname{diag}(\lambda(x), \sigma_2\lambda(x), \dots, \sigma_n\lambda(x)), \tag{6}$$

so that $\phi \sim_{\bar{K}} \psi$. By Lemma 2.4, $\phi \sim \psi$ and $V \cong W$.

Part 2. Let $\lambda : K \to \bar{K}$ be an embedding with $\lambda^G \in \Lambda(K)$. We shall construct a simple two-sided vector space $V(\lambda) = {}_1K^n_\phi$ from λ, such that $v = (\alpha_1, \dots, \alpha_n) \in K(\lambda)^n$ is a common eigenvector for $\operatorname{im}\phi$, with corresponding eigenvalue λ. Retaining the above notation, we define a map $\phi = (\phi_{ij}) : K \to M_n(K)$ by

$$\phi_{ij}(x) = \sum_{k=1}^{n} \beta_{jki}\lambda_k(x). \tag{7}$$

Part 2, Step 1. We prove that, for all $\sigma \in G$ and $x \in K$, $\sigma(v)$ is an eigenvector for $\phi(x)$ with eigenvalue $\sigma\lambda(x)$. We have $\sigma(v) = (\sigma(\alpha_1), \dots, \sigma(\alpha_n))$ and $\sigma\lambda(x) = \sum_{i=1}^n \lambda_i(x)\sigma(\alpha_i)$. On the one hand,

$$\begin{aligned}\sigma(v)\phi(x) &= (\sigma(\alpha_1), \dots, \sigma(\alpha_n))\phi(x)\\ &= \big(\sum_i \phi_{i1}(x)\sigma(\alpha_i), \dots, \sum_i \phi_{in}(x)\sigma(\alpha_i)\big)\\ &= \big(\sum_{i,k} \beta_{1ki}\lambda_k(x)\sigma(\alpha_i), \dots, \sum_{i,k} \beta_{nki}\lambda_k(x)\sigma(\alpha_i)\big).\end{aligned} \tag{8}$$

On the other hand,

$$\begin{aligned}\sigma\lambda(x)\sigma(v) &= \big(\sum_k \lambda_k(x)\sigma(\alpha_k)\big)(\sigma(\alpha_1), \dots, \sigma(\alpha_n))\\ &= \big(\sum_k \lambda_k(x)\sigma(\alpha_k\alpha_1), \dots, \sum_k \lambda_k(x)\sigma(\alpha_k\alpha_n)\big)\\ &= \big(\sum_{i,k} \lambda_k(x)\beta_{k1i}\sigma(\alpha_i), \dots, \sum_{i,k} \lambda_k(x)\beta_{kni}\sigma(\alpha_i)\big)\end{aligned} \tag{9}$$

Comparing coordinates and using the identity $\beta_{pqr} = \beta_{qpr}$ for all p, q, r gives the result.

Part 2, Step 2. We show ϕ is a homomorphism. Since each λ_k is an additive function it is clear that ϕ is additive. To see that ϕ is multiplicative, write out $\lambda^G = \{\sigma_1\lambda, \dots, \sigma_n\lambda\}$ (where σ_1 is the identity). Then $\{\sigma_1(v), \dots, \sigma_n(v)\}$ is a basis for $\bar{K}^n$, and for all $x, y \in K$, we have

$$\begin{aligned}\sigma_i(v)\phi(x)\phi(y) = \sigma_i\lambda(x)\sigma_i(v)\phi(y) &= \sigma_i\lambda(x)\sigma_i\lambda(y)\sigma_i(v)\\ &= \sigma_i\lambda(xy)\sigma_i(v) = \sigma_i(v)\phi(xy).\end{aligned} \tag{10}$$

This shows that $\phi(x)\phi(y)$ and $\phi(xy)$ act as the same linear transformation on each $\sigma_i(v)$. Since the $\sigma_i(v)$ form a basis for $\bar{K}^n$, we have that $\phi(x)\phi(y) = \phi(xy)$ for all $x, y \in K$.

Part 2, Step 3. Since ϕ is a homomorphism, we can define the two-sided vector space $V(\lambda) = {}_1K^n_\phi$. We prove $V(\lambda)$ is simple. Suppose that W is a simple sub-bimodule of $V(\lambda)$ with $\dim W = m$, and fix a left basis for $V(\lambda)$ containing a left basis for W. Then, relative to this basis, we have $V(\lambda) \cong {}_1K^n_\psi$, where $\psi = \begin{pmatrix}\psi_1 & \theta\\ 0 & \psi_2\end{pmatrix}$ and $W \cong {}_1K^m_{\psi_2}$. Since W is simple, there is a unique orbit $\mu^G = \{\mu_1, \dots, \mu_m\} \in \Lambda(K)$ with $\psi_2 \sim_{\bar{K}} \operatorname{diag}(\mu_1, \dots, \mu_m)$. On the other hand, we have by definition of $V(\lambda)$ that $\phi \sim_{\bar{K}} \operatorname{diag}(\lambda, \sigma_2\lambda, \dots, \sigma_n\lambda)$; since $\phi \sim \psi$ we see that $\mu_1 = \sigma_j\lambda$ for some j. Hence $\mu^G = \lambda^G$ and $W = V(\lambda)$ since Φ is injective. $\square$

To complete the proof of Theorem 3.2, we need to compute $\mathrm{End}(V(\lambda))$.

Proposition 3.6. $\mathrm{End}(V(\lambda)) \cong K(\lambda)$.

Proof. Let $|\lambda^G| = n$. We first note that $\mathrm{End}(V(\lambda))$ can be made into a left vector space over K by defining $(xf)(v) = xf(v)$ for $x \in K$, $v \in V(\lambda)$, and $f \in \mathrm{End}(V(\lambda))$. Also, since $V(\lambda)$ is a simple bimodule, it is generated as a bimodule by a single element w. If $\{f_1, \ldots, f_{n+1}\}$ is a subset of $\mathrm{End}(V(\lambda))$, then $\{f_1(w), \ldots, f_{n+1}(w)\}$ are necessarily linearly dependent in $V(\lambda)$; hence there exist $x_i \in K$ such that the endomorphism $\sum_{i=1}^{n+1} x_i f_i$ acts as 0 on w. Since w generates $V(\lambda)$ we see that $\sum_{i=1}^{n+1} x_i f_i = 0$ and so $\dim \mathrm{End}(V(\lambda)) \leq n$.

Fix an isomorphism $V(\lambda) \cong {}_1K^n_{\phi}$, and let $\{e_1, \ldots, e_n\}$ be the standard basis for K^n. Given $f \in \mathrm{End}(V(\lambda))$, we can write $f(e_i) = \sum_j f_{ji} e_j$, where $f_{ji} \in K$. Then the map $f \mapsto M(f) = (f_{ij})$ allows us to realize each $f \in \mathrm{End}(V(\lambda))$ as right multiplication by the matrix $M(f) \in M_n(K)$. The fact that f is a bimodule endomorphism is equivalent to $M(f)$ commuting with $\phi(x)$ for all $x \in K$. Conversely, if $M \in M_n(K)$ with $M = (m_{ij})$ and if M commutes with $\phi(x)$, the rule $e_i \mapsto \sum_j m_{ji} e_j$ makes M an element of $\mathrm{End}(V(\lambda))$.

For each $p \leq n$, let $M(p)$ be the matrix given by $M(p)_{ij} = \beta_{pji}$. We prove that $M(p) \in \mathrm{End}(V(\lambda))$. If $v = (\alpha_1, \ldots, \alpha_n)$ as in (3), then one calculates that

$$\sigma(v)M(p) = (\sigma(\alpha_1), \ldots, \sigma(\alpha_n))(\beta_{pji}) = \big(\sum_j \beta_{p1j}\sigma(\alpha_j), \ldots, \sum_j \beta_{pnj}\sigma(\alpha_j)\big) \tag{11}$$

for all $\sigma \in \mathrm{Aut}(\bar{K}/K)$. On the other hand,

$$\sigma(\alpha_p)\sigma(\alpha_i) = \sigma(\alpha_p\alpha_i) = \sigma(\sum_j \beta_{pij}\alpha_j) = \sum_j \beta_{pij}\sigma(\alpha_j). \tag{12}$$

Hence we see that the i-th component of $\sigma(v)M(p)$ is $\sigma(\alpha_p)\sigma(\alpha_i)$, and we conclude that $\sigma(v)$ is an eigenvector for $M(p)$ with eigenvalue $\sigma(\alpha_p)$; in particular, we see that $\sigma(v)M(p)M(q) = \sigma(v)M(q)M(p)$ for all $p, q \leq n$ and $\sigma \in \mathrm{Aut}(\bar{K}/K)$. Since $\{v, \sigma_2(v), \ldots, \sigma_n(v)\}$ is a basis for $\bar{K}^n$, we conclude that in fact $M(p)$ and $M(q)$ commute for all p, q. Finally, since $\sigma_i(v)$ is a common eigenvector for $\phi(x)$ and $M(p)$ for all $p \leq n$ and $x \in K$, we see that $M(p)$ and $\phi(x)$ commute. Therefore, $\{M(1), \ldots, M(n)\}$ are pairwise commuting, K-linearly independent elements of $\mathrm{End}(V(\lambda))$. Since $\dim \mathrm{End}(V(\lambda)) \leq n$, we conclude that $\mathrm{End}(V(\lambda)) \cong K\{M(1), \ldots, M(n)\}$; one checks easily that the map $M(p) \mapsto \alpha_p$ gives the desired ring isomorphism $\mathrm{End}(V(\lambda)) \cong K(\lambda)$. ☐

We illustrate Theorem 3.2 with several examples.

Example 3.7. Suppose that there exists $\lambda \in \mathrm{Emb}(K)$ with $|\lambda^G| = 2$. Then $K(\lambda)$ is a degree 2 extension of K, and so $K(\lambda) = K(\sqrt{m})$ for some $m \in K$. Then $\mathrm{Aut}(K(\sqrt{m})/K)$ is generated by σ, where $\sigma(\sqrt{m}) = -\sqrt{m}$, and $\lambda^G = \{\lambda, \sigma\lambda\}$.

Using $\{1, \sqrt{m}\}$ as a K-basis for $\sqrt{m}$, we can write $\lambda(x) = \lambda_1(x) + \lambda_2(x)\sqrt{m}$. If we write out the matrix $(\phi_{ij}(x))$, we see that

$$\phi(x) = \begin{pmatrix} \lambda_1(x) & m\lambda_2(x) \\ \lambda_2(x) & \lambda_1(x) \end{pmatrix}, \tag{13}$$

and that

$$(1, \sqrt{m}) \begin{pmatrix} \lambda_1(x) & m\lambda_2(x) \\ \lambda_2(x) & \lambda_1(x) \end{pmatrix} = (\lambda(x), \lambda(x)\sqrt{m}) = \lambda(x)(1, \sqrt{m}).$$

Moreover, The fact that ϕ is a homomorphism gives the formulas

$$\lambda_1(xy) = \lambda_1(x)\lambda_1(y) + m\lambda_2(x)\lambda_2(y)$$
$$\lambda_2(xy) = \lambda_1(x)\lambda_2(y) + \lambda_1(y)\lambda_2(x).$$

Thus we recover [5, Theorem 1.3(iii)] as a special case of Theorem 3.2. □

Example 3.8. Let K be a field of characteristic different from 3, and suppose there exists $\lambda \in \mathrm{Emb}(K)$ such that $|\lambda^G| = 3$. Then $[K(\lambda) : K] = 3$, so that $K(\lambda) = K(\gamma)$, where γ is the root of an irreducible polynomial $x^3 + bx + c$ with $b, c \in K$. Thus, $\{1, \gamma, \gamma^2\}$ is a basis of $K(\lambda)/K$. In this basis, we find that $(\phi_{ij}(x))$ is the matrix

$$\begin{pmatrix} \lambda_0(x) & -c\lambda_2(x) & -c\lambda_1(x) \\ \lambda_1(x) & \lambda_0(x) - b\lambda_2(x) & -b\lambda_1(x) - c\lambda_2(x) \\ \lambda_2(x) & \lambda_1(x) & \lambda_0(x) - b\lambda_2(x) \end{pmatrix}$$

where the functions λ_0, λ_1 and λ_2 satisfy the relations

$$\lambda_0(xy) = \lambda_0(x)\lambda_0(y) - c\lambda_2(x)\lambda_1(y) - c\lambda_1(x)\lambda_2(y)$$
$$\lambda_1(xy) = \lambda_1(x)\lambda_0(y) + (\lambda_0(x) + b\lambda_2(x))\lambda_1(y) - (b\lambda_1(x) + c\lambda_2(x))\lambda_2(y))$$
$$\lambda_2(xy) = \lambda_2(x)\lambda_0(y) + \lambda_1(x)\lambda_1(y) + (\lambda_0(x) - b\lambda_2(x))\lambda_2(y).$$

□

Example 3.9. Suppose $p \geq 3$ is prime, $\rho = \sqrt[p]{2}$, ζ is a primitive p-th root of unity, $k = \mathbb{Q}$ and $K = \mathbb{Q}(\rho)$. Then $K(\zeta)$ is the Galois closure of $K/\mathbb{Q}$, with $\mathrm{Aut}(K(\zeta)/K) = \{\sigma_i : 1 \leq i \leq p-1\}$, where $\sigma_i(\zeta) = \zeta^i$. If we let $\lambda : K \to \bar{K}$ be the embedding that takes ρ to $\zeta\rho$, then $\mathrm{Emb}(K) = \{\mathrm{Id}_K\} \cup \{\sigma_i\lambda : 1 \leq i \leq p-1\}$. Hence $\Lambda(K)$ consists of the two orbits $\{\mathrm{Id}_K\}$ and $\lambda^G = \{\sigma_i\lambda : 1 \leq i \leq p-1\}$, and so there are up to isomorphism two simples in $\mathsf{Vect}(K)$: the trivial simple bimodule K corresponding to $\{\mathrm{Id}_K\}$, and a $p-1$-dimensional simple corresponding to λ^G.

We now construct the matrix homomorphism $\phi : K \to M_{p-1}(K)$ representing the $p-1$-dimensional simple as in (7). First, taking $\{1, \zeta, \ldots, \zeta^{p-2}\}$ as a basis of $K(\zeta)/K$ and letting $\alpha_i = \zeta^i$ for $0 \leq i \leq p-2$ (we have shifted our indices for ease of computation), we compute the constants β_{jki}: if $j + k \neq p-1$, then

$$\alpha_j\alpha_k = \zeta^j\zeta^k = \zeta^{j+k} = \alpha_{j+k},$$

where the superscripts and subscripts are taken modulo p. Therefore, when $j + k \neq p-1$, $\beta_{jki} = 1$ if and only if $i \equiv j + k \pmod p$, and $\beta_{jki} = 0$ otherwise.

If $j + k = p - 1$, then

$$\alpha_j\alpha_k = \zeta^{p-1} = -1 - \zeta - \cdots - \zeta^{p-2} = -\alpha_0 - \alpha_1 - \cdots - \alpha_{p-2}.$$

Therefore, when $j + k = p - 1$, $\beta_{jki} = -1$ for all $0 \leq i \leq p-2$.

Thus,

(1) if $j = 0$ then $j + k \neq p - 1$, so $\beta_{0ki} = \delta_{ki}$, and
(2) if $j \neq 0$, either
  (a) $k = p - 1 - j$, in which case $\beta_{j,p-1-j,i} = -1$ for all i or
  (b) $k \neq p - 1 - j$, in which case $\beta_{j,i-j,i} = 1$ for all $i \neq j - 1$ (where subscripts are taken modulo p) and $\beta_{jki} = 0$ otherwise.

Next, we write

$$\lambda(x) = \lambda_0(x) + \lambda_1(x)\zeta + \cdots + \lambda_{p-2}(x)\zeta^{p-2}$$

and determine the functions $\lambda_i(x)$, $0 \le i \le p-2$. If $x \in K$, we may write $x = \sum_{l=0}^{p-1} a_l \rho^l$ with $a_0, \ldots, a_{p-1} \in \mathbb{Q}$. It is then easy to see that

$$\lambda_i(\sum_{l=0}^{p-1} a_l \rho^l) = a_i \rho^i - a_{p-1}\rho^{p-1}$$

for $0 \le i \le p-2$.

Using the formula $\phi_{ij}(x) = \sum_{k=0}^{p-2} \beta_{jki}\lambda_k(x)$, we may deduce that $\phi(x)$ is the matrix

$$\begin{pmatrix} \lambda_0(x) & -\lambda_1(x) \\ \lambda_1(x) & -\lambda_1(x)+\lambda_0(x) \end{pmatrix} \tag{14}$$

when $p = 3$ and $\phi(x)$ is the matrix

$$\begin{pmatrix} \lambda_0(x) & -\lambda_{p-2}(x) & -\lambda_{p-3}(x)+\lambda_{p-2}(x) & \cdots & -\lambda_1(x)+\lambda_2(x) \\ \lambda_1(x) & -\lambda_{p-2}(x)+\lambda_0(x) & -\lambda_{p-3}(x) & \cdots & -\lambda_1(x)+\lambda_3(x) \\ \lambda_2(x) & -\lambda_{p-2}(x)+\lambda_1(x) & -\lambda_{p-3}(x)+\lambda_0(x) & \cdots & -\lambda_1(x)+\lambda_4(x) \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_{p-3}(x) & -\lambda_{p-2}(x)+\lambda_{p-4}(x) & -\lambda_{p-3}(x)+\lambda_{p-5}(x) & \cdots & -\lambda_1(x) \\ \lambda_{p-2}(x) & -\lambda_{p-2}(x)+\lambda_{p-3}(x) & -\lambda_{p-3}+\lambda_{p-4}(x) & \cdots & -\lambda_1(x)+\lambda_0(x) \end{pmatrix}$$

when $p \ge 5$.

Had we chosen a different basis for $K(\lambda)$ over K, then $\phi(x)$ would have a different form. For example, when $p = 3$, then $K(\zeta) = K(\sqrt{-3})$. If we use $\{1, \sqrt{-3}\}$ as a basis for $K(\zeta)$ over K, then we find that $\phi(x)$ takes the form (13). □

We conclude this section by noting that there are no nontrivial extensions between nonisomorphic simple bimodules. The result is probably well known, but we were unable to find a reference.

Proposition 3.10. $\mathrm{Ext}^1_{K^e}(V, W) = 0$ *for nonisomorphic simple bimodules* V, W.

Proof. Suppose that V and W are nonisomorphic simple bimodules. Let $\Phi([V]) = \lambda^G$ and $\Phi([W]) = \mu^G$, and fix isomorphisms $V \cong {}_1K^m_\phi$ and $W \cong {}_1K^n_\psi$. If U is an extension of V by W, then there is a basis for K^{m+n} such that $U \cong {}_1K^{m+n}_\eta$, where $\eta = \begin{pmatrix} \psi & \theta \\ 0 & \phi \end{pmatrix}$ for some $\theta : K \to M_{n\times m}(K)$. Enumerate the elements of λ^G and μ^G as $\{\lambda_1, \ldots, \lambda_m\}$ and $\{\mu_1, \ldots, \mu_n\}$, respectively. Then we have that

$$\begin{pmatrix} \phi & \theta \\ 0 & \psi \end{pmatrix} \sim_{\bar{K}} \mathrm{diag}(\lambda_1, \ldots, \lambda_m, \mu_1 \ldots, \mu_n) \sim_{\bar{K}} \begin{pmatrix} \phi & 0 \\ 0 & \psi \end{pmatrix}. \tag{15}$$

It follows by Lemma 2.4 that $U \cong V \oplus W$. □

4. Algebraic K-theory of $\mathsf{Vect}(K)$

We shall denote by $K^B_i(K)$ the Quillen K-theory of $\mathsf{Vect}(K)$ [6] (the superscript stands for "bimodule"). The description of the simples in $\mathsf{Vect}(K)$ in Section 3 and the Devissage Theorem [6, Corollary 5.1] immediately yield the following result.

Theorem 4.1. *For all* $i \ge 0$, *there is an isomorphism of abelian groups*

$$K^B_i(K) \cong \bigoplus_{\lambda^G \in \Lambda(K)} K_i(K(\lambda)). \tag{16}$$

The Grothendieck group $K_0^B(K)$ can be made into a commutative ring by defining multiplication via the tensor product. Thus, if V and W are simple bimodules in $\mathsf{Vect}(K)$, we define $[V]\cdot[W]=[V\otimes W]$ in $K_0^B(K)$. (Here and below $\otimes$ denotes the tensor product over K.) In particular, $[V]\cdot[W]=\sum_{i=1}^t [V_i]$, where $V_1,\ldots,V_t$ are the composition factors of $V\otimes W$. There is an especially nice description of $K_0^B(K)$ when $\mathrm{Emb}(K)=\mathrm{Aut}(K)$; this will happen for instance if K is a normal algebraic extension of the centralizing subfield k.

Proposition 4.2. *If K is a field with $\mathrm{Emb}(K)=\mathrm{Aut}(K)$, then there is a ring isomorphism $K_0^B(K)\cong\mathbb{Z}[\mathrm{Aut}(K)]$.*

Proof. Each simple bimodule in $\mathsf{Vect}(K)$ is isomorphic to ${}_1K_\phi$ for some $\phi\in\mathrm{Aut}(K)$. The map $[{}_1K_\phi]\mapsto\phi$ then gives an isomorphism between the abelian groups $K_0^B(K)$ and $\mathbb{Z}[\mathrm{Aut}(K)]$. Moreover, an elementary calculation shows that ${}_1K_\phi\otimes{}_1K_\psi\cong{}_1K_{\phi\psi}$. From this it follows readily that the above map is actually a ring isomorphism. □

In order to describe the ring structure of $K_0^B(K)$ for a general field K, we shall need to introduce some notation. Identifying the K-algebras $M_m(K)\otimes M_n(K)$ and $M_{mn}(K)$, we introduce multi-index notation to refer to the coordinates of $M_{mn}(K)$ as follows. Order the pairs (i,j) with $1\le i\le m$ and $1\le j\le n$ lexicographically; then there is a bijection between these pairs and $\{1,\ldots,mn\}$. We shall write $A_{(i_1,i_2),(j_1,j_2)}$ for the entry of $A\in M_{mn}(K)$ whose row corresponds to (i_1,i_2) and whose column corresponds to (j_1,j_2) under this bijection. The reason for adopting this notation is that, if $A=(a_{ij})\in M_m(K)$ and $B=(b_{ij})\in M_n(K)$, then $(A\otimes B)_{(i_1,i_2),(j_1,j_2)}=a_{i_1j_1}b_{i_2j_2}$, where $A\otimes B$ is the Kronecker product of A and B.

The following is a variant of the Kronecker product for functions.

Definition 4.3. Let $\phi=(\phi_{ij}):K\to M_m(K)$ and $\psi=(\psi_{ij}):K\to M_n(K)$ be functions. Then we define their *Kronecker composition* $\phi\otimes\psi:K\to M_{mn}(K)$ by the rule $(\phi\otimes\psi)_{(i_1,i_2),(j_1,j_2)}=\phi_{i_1j_1}\circ\psi_{i_2j_2}$. Similarly, if $A=(a_{ij})\in M_n(K)$, then we define $\phi\otimes A\in M_{mn}(K)$ to be the matrix given by $(\phi\otimes A)_{(i_1,i_2),(j_1,j_2)}=\phi_{i_1j_1}(a_{i_2j_2})$. Note that if $x\in K$, then $(\phi\otimes\psi)(x)=\phi\otimes(\psi(x))$, so that the two definitions are consistent with each other. Finally, if $B\in M_m(K)$, then we define the functions ϕB and $B\phi$ by $(\phi B)(x)=\phi(x)B$ and $(B\phi)(x)=B\phi(x)$, respectively.

The utility of the Kronecker composition in understanding tensor products of bimodules is revealed in the following lemma. In particular, it implies that when $\phi:K\to M_m(K)$ and $\psi:K\to M_n(K)$ are homomorphisms, so too is $\phi\otimes\psi:K\to M_{mn}(K)$.

Lemma 4.4. *Given homomorphisms $\phi:K\to M_m(K)$ and $\psi:K\to M_n(K)$, we have ${}_1K^m_\phi\otimes{}_1K^n_\psi\cong{}_1K^{mn}_{\phi\otimes\psi}$.*

Proof. Let $\{e_1,\ldots,e_m\}$ and $\{f_1,\ldots,f_n\}$ be the standard left bases for K^m and K^n, respectively. If we let $e_{(i,j)}=e_i\otimes f_j$, then $\{e_{(i,j)}:1\le i\le m,\ 1\le j\le n\}$ gives a left basis for K^{mn}. We compute the right action of K on K^{mn} under this basis:

$$\begin{aligned}
e_{(i,j)} \cdot x &= (e_i \otimes f_j) \cdot x = e_i \otimes f_j \psi(x) = e_i \otimes \sum_{l=1}^{n} \psi_{jl}(x) f_l \\
&= \sum_{l=1}^{n} e_i \cdot \psi_{jl}(x) \otimes f_l = \sum_{l=1}^{n} e_i \phi(\psi_{jl}(x)) \otimes f_l \\
&= \sum_{l=1}^{n} \sum_{k=1}^{m} \phi_{ik}(\psi_{jl}(x)) e_k \otimes f_l = \sum_{(k,l)} \phi_{ik} \circ \psi_{jl}(x) e_{(k,l)}. \\
&= \sum_{(k,l)} (\phi \otimes \psi)_{(i,j),(k,l)}(x) e_{(k,l)} = e_{(i,j)}(\phi \otimes \psi)(x).
\end{aligned} \tag{17}$$

□

Lemma 4.5. *Let $\phi : K \to M_m(K)$ and $\psi : K \to M_n(K)$ be homomorphisms and let $A = (a_{ij}) \in M_m(K)$, $B = (b_{ij}), C = (c_{ij}) \in M_n(K)$. Then the following hold:*

1. *$(\phi \otimes B)(\phi \otimes C) = \phi \otimes BC$.*
2. *$(A \otimes I_n)(\phi \otimes B) = (A\phi) \otimes B$ and $(\phi \otimes B)(A \otimes I_n) = (\phi A) \otimes B$, where I_n is the $n \times n$ identity matrix.*
3. *If $\phi \sim \phi'$ and $\psi \sim \psi'$, then $\phi \otimes \psi \sim \phi' \otimes \psi'$.*

Proof. (1) We compute the $(i_1, i_2), (j_1, j_2)$ component of $(\phi \otimes B)(\phi \otimes C)$:

$$\begin{aligned}
(\phi \otimes B)(\phi \otimes C)_{(i_1,i_2),(j_1,j_2)} &= \sum_{(k,l)} (\phi \otimes B)_{(i_1,i_2),(k,l)} (\phi \otimes C)_{(k,l),(j_1,j_2)} \\
&= \sum_{(k,l)} \phi_{i_1 k}(b_{i_2 l}) \phi_{k j_1}(c_{l j_2}) \\
&= \sum_{l} \phi_{i_1 j_1}(b_{i_2 l} c_{l j_2}) \quad (\phi \text{ is a homomorphism}) \\
&= \phi_{i_1 j_1}((BC)_{i_2 j_2}) = (\phi \otimes BC)_{(i_1,i_2),(j_1,j_2)}.
\end{aligned} \tag{18}$$

(2) Again, the proof is a computation. We show the first equality and leave the second to the reader.

$$\begin{aligned}
(A \otimes I_n)(\phi \otimes B)_{(i_1,i_2),(j_1,j_2)} &= \sum_{(k,l)} (A \otimes I_n)_{(i_1,i_2),(k,l)} (\phi \otimes B)_{(k,l),(j_1,j_2)} \\
&= \sum_{(k,l)} a_{i_1 k} (I_n)_{i_2 l} \phi_{k j_1}(b_{l j_2}).
\end{aligned} \tag{19}$$

The only nonzero term in the sum occurs when $l = i_2$, because of the $(I_n)_{i_2 l}$ term. Hence the above sum collapses to

$$\sum_{k} a_{i_1 k} \phi_{k j_1}(b_{i_2 j_2}) = (A\phi)_{i_1 j_1}(b_{i_2 j_2}) = (A\phi \otimes B)_{(i_1,i_2),(j_1,j_2)}. \tag{20}$$

(3) First, suppose that $B \in M_n(K)$ is invertible. Then by part (1), we have $(\phi \otimes B)(\phi \otimes B^{-1}) = \phi \otimes BB^{-1} = \phi \otimes I_n = I_{mn}$. Thus $\phi \otimes B$ is invertible, with inverse $\phi \otimes B^{-1}$. Now, suppose that $B\psi(x)B^{-1} = \psi'(x)$ for all $x \in K$, and that $A\phi(x)A^{-1} = \phi'(x)$ for all $x \in K$. Then, for all $x \in K$, we have

$$(A \otimes I_n)(\phi \otimes B)(\phi \otimes \psi(x))(\phi \otimes B^{-1})(A^{-1} \otimes I_n) = (\phi' \otimes \psi')(x) \tag{21}$$

by parts (1) and (2) above. Hence $\phi \otimes \psi \sim \phi' \otimes \psi'$. □

Any embedding λ of K into $\bar{K}$ can be lifted to an automorphism $\bar{\lambda}$ of $\bar{K}$, such that $\bar{\lambda}|_K = \lambda$. The following lemma extends this to certain homomorphisms $\phi : K \to M_n(K)$.

Lemma 4.6. *If $\phi : K \to M_n(K)$ represents a simple bimodule, then there exists a homomorphism $\bar{\phi} : \bar{K} \to M_n(\bar{K})$ such that $\bar{\phi}|_K = \phi$.*

Proof. Write ${}_1K_\phi^m \cong V(\lambda)$ for some $\lambda^G \in \Lambda(K)$, and write $\lambda^G = \{\lambda_1, \dots, \lambda_m\}$. Viewing ϕ as a function from K to $M_m(\bar{K})$, there exists $P \in GL_m(\bar{K})$ such that $\phi(x) = P\,\mathrm{diag}(\lambda_1(x), \dots, \lambda_m(x))P^{-1}$ for all $x \in K$. Lift each λ_i to $\bar{\lambda}_i : \bar{K} \to \bar{K}$, and define $\bar{\phi}$ by the formula

$$\bar{\phi}(x) = P\,\mathrm{diag}(\bar{\lambda}_1(x), \dots, \bar{\lambda}_m(x))P^{-1}. \tag{22}$$

Then one easily checks that $\bar{\phi}$ is a lift of ϕ. □

The above result obviously extends to semisimple bimodules by induction, but we will only need to apply it in the case where V is simple.

Theorem 4.7. *Let $\lambda^G, \mu^G \in \Lambda(K)$. Then $V(\lambda) \otimes V(\mu)$ is semisimple.*

Proof. If K is finite, then each of λ and μ is an automorphism of K, and $V(\lambda) \otimes V(\mu) \cong V(\lambda\mu)$ is simple. So we may assume that K is infinite. Enumerate the elements of λ^G and μ^G as $\{\lambda_1, \dots, \lambda_m\}$ and $\{\mu_1, \dots, \mu_n\}$ respectively, and let $\bar{\lambda}_i$ and $\bar{\mu}_j$ be lifts of λ_i and μ_j to automorphisms of $\bar{K}$. If we write $V(\lambda) \cong {}_1K_\phi^m$ and $V(\mu) \cong {}_1K_\psi^n$, then the previous lemma shows that there are lifts $\bar{\phi} : \bar{K} \to M_m(\bar{K})$ and $\bar{\psi} : \bar{K} \to M_n(\bar{K})$, such that $\bar{\phi} \sim \mathrm{diag}(\bar{\lambda}_1, \dots, \bar{\lambda}_m)$ and $\bar{\psi} \sim \mathrm{diag}(\bar{\mu}_1, \dots, \bar{\mu}_n)$. It follows from Lemma 4.5 and an elementary calculation that $\bar{\phi} \otimes \bar{\psi} \sim \mathrm{diag}(\bar{\lambda}_i\bar{\mu}_j : 1 \le i \le m,\ 1 \le j \le n)$.

For each pair (i, j), let $\nu_{ij} = \bar{\lambda}_i\bar{\mu}_j|_K$. Then $\nu_{ij} \in \mathrm{Emb}(K)$ and $\nu_{ij}^G \in \Lambda(K)$. Moreover, an easy calculation shows that $\bar{\phi} \otimes \bar{\psi}|_K = \phi \otimes \psi$, and from this we conclude that $\phi \otimes \psi \sim_{\bar{K}} \mathrm{diag}(\nu_{ij})$. Partition the multiset $\{\nu_{ij}\}$ into a union of disjoint orbits, counting multiplicities, say $\{\nu_{ij}\} = \bigcup_{k=1}^t (m_k)\nu_k^G$, where $(m_k)\nu_k^G$ means m_k copies of ν_k^G. Let $V = \oplus_{k=1}^t V(\nu_k)^{(m_k)}$ and write $V \cong {}_1K_\theta^{mn}$ for some θ. Then by construction we have that $\phi \otimes \psi \sim_{\bar{K}} \theta$; by Lemma 2.4 $\phi \otimes \psi \sim \theta$, so that $V(\lambda) \otimes V(\mu) \cong \oplus_{k=1}^t V(\nu_k)^{(m_k)}$ is semisimple. □

The above theorem yields a presentation for the ring $K_0^B(K)$ by generators and relations. We distinguish between the trivial simple bimodule K which corresponds to $\{\mathrm{Id}_K\} \in \Lambda(K)$ and acts as the identity of $K_0^B(K)$, and the nontrivial simple bimodules $\{V(\lambda) : \lambda^G \ne \{\mathrm{Id}_K\}\}$.

Corollary 4.8. *Write $\Lambda(K) = \{\mathrm{Id}_K\} \cup (\bigcup_{i \in I} \lambda_i^G)$ as a union of disjoint orbits, and for each pair i, j, write $V(\lambda_i) \otimes V(\lambda_j) \cong K^{(\alpha_{ij})} \oplus (\bigoplus_{l \in I} V(\lambda_l)^{(\alpha_{ijl})})$ for nonnegative integers $\alpha_{ij}, \alpha_{ijl}$. Then $K_0^B(K)$ is isomorphic to the quotient of $\mathbb{Z}\langle\{x_i : i \in I\}\rangle$ by the ideal I generated by $\{x_i x_j - (\sum_{l \in I} \alpha_{ijl} x_l + \alpha_{ij}) : i, j \in I\}$.*

The following example illustrates how one can use Theorem 4.7 to find an explicit presentation for $K_0^B(K)$.

Example 4.9. Let p be an odd prime and let $K = \mathbb{Q}(\rho)$, where ρ is a real p-th root of 2. As in Example 3.9, $\mathrm{Emb}(K)$ is partitioned into two orbits: $\mathrm{Emb}(K) = \{\mathrm{Id}_K\} \cup \lambda^G$, where λ is the embedding defined by $\lambda(\rho) = \zeta\rho$. Now, $\mathrm{Aut}(K(\zeta)/K)$ is cyclic of order $p - 1$; let σ be a generator for $\mathrm{Aut}(K(\zeta)/K)$. To be precise, let

$\sigma(\zeta) = \zeta^n$, where n is a multiplicative generator for $(\mathbb{Z}/p\mathbb{Z})^*$. There are obvious lifts of λ and σ to automorphisms of $\bar{K}$; we abuse notation and denote these lifts by λ and σ as well.

There are exactly two simple bimodules up to isomorphism: The trivial bimodule K, and the $p-1$-dimensional bimodule $V(\lambda)$. In order to calculate the ring structure on $K_0^B(K)$, we must decompose $V(\lambda) \otimes V(\lambda)$ as a direct sum of simples.

If we write $V(\lambda) \cong {}_1K_\phi^{p-1}$, then $\phi \sim_{\bar{K}} \operatorname{diag}(\sigma^i\lambda : 0 \leq i \leq p-2)$. Hence $\phi \otimes \phi \sim_{\bar{K}} \operatorname{diag}(\sigma^i\lambda\sigma^j\lambda : 0 \leq i, j \leq p-2)$. So, we must count the number of times that $\sigma^i\lambda\sigma^j\lambda|_K = \mathrm{Id}_K$.

We compute:

$$\sigma^i\lambda\sigma^j\lambda(\rho) = \sigma^i\lambda\sigma^j(\zeta\rho) = \sigma^i\lambda(\zeta^{n^j}\rho) = \sigma^i(\zeta^{n^j+1}\rho) = \zeta^{n^i(n^j+1)}\rho.$$

So, we must have $n^i(n^j+1) \equiv 0 \pmod{p}$. Since $(n, p) = 1$, this only happens when $n^j+1 \equiv 0 \pmod{p}$, and since n is a multiplicative generator for $(\mathbb{Z}/p\mathbb{Z})^*$, this only happens for $j = (p-1)/2$. For this value of j, we see that $\sigma^i\lambda\sigma^{(p-1)/2}\lambda|_K = \mathrm{Id}_K$ for *all* i; in particular, there are exactly $p-1$ copies of the trivial bimodule as a summand of $V(\lambda) \otimes V(\lambda)$.

The rest is a dimension count: Since $\dim V(\lambda) \otimes V(\lambda) = (p-1)^2$ and $V(\lambda) \otimes V(\lambda) \cong K^{(p-1)} \oplus V(\lambda)^{(t)}$, it follows that $t = p-2$; i.e. $V(\lambda) \otimes V(\lambda) \cong K^{(p-1)} \oplus V(\lambda)^{(p-2)}$.

From this we conclude that $K_0^B(K) \cong \mathbb{Z}[x]/(x^2 - (p-2)x - (p-1))$. □

We conclude this section with a brief discussion of an alternative, "naive" approach to the Grothendieck ring of $\mathsf{Vect}(K)$. Namely, one could consider the free abelian group on isomorphism classes in $\mathsf{Vect}(K)$, modulo only those relations induced by direct sums (instead of all exact sequences). We denote this ring by $K_0^{\oplus}(K)$. Our aim is to show that, while $K_0^B(K)$ is computable in many cases, $K_0^{\oplus}(K)$ is an intractable object of study. We begin with a definition.

Definition 4.10. A *higher k-derivation of order m* (or an *m-derivation*) on K is a sequence of k-linear maps $\mathbf{d} = \{d_0, d_1, \ldots, d_m\}$, such that $d_l(xy) = \sum_{i+j=l} d_i(x)d_j(y)$ for all $x, y \in K$. (In particular $d_0 : K \to K$ is an endomorphism and d_1 is a d_0-derivation.) We denote the set of all n-derivations by $HS_n(K)$, and the set of all higher derivations (of all orders) by $HS(K)$. We refer the reader to [4, Section 27] for more information on higher derivations. (Note that our definition is slightly more general, in that [4] assumes that $d_0 = \mathrm{Id}_K$).

Note that $HS(K)$ can be made into an abelian semigroup with identity as follows: Given $\mathbf{d} = \{d_0, \ldots, d_m\}$ and $\mathbf{d}' = \{d_0', \ldots, d_n'\}$, we define $\mathbf{d} \cdot \mathbf{d}' = \{\delta_0, \ldots, \delta_{m+n}\}$, where $\delta_l = \sum_{i+j=l} d_i d_j'$. (Here we set $d_i = 0$ for $i > m$ and $d_j' = 0$ for $j > n$.) The above operation actually makes $HS(K)$ a group, but we will not need this fact below.

Given $\mathbf{d} = \{d_0, d_1, \ldots, d_m\}$, we define a map $\phi(\mathbf{d}) : K \to M_{m+1}(K)$ by

$$\phi(\mathbf{d})(x) = \begin{pmatrix} d_0(x) & d_1(x) & \ldots & d_m(x) \\ 0 & d_0(x) & \ddots & \vdots \\ \vdots & \ddots & \ddots & d_1(x) \\ 0 & \ldots & 0 & d_0(x) \end{pmatrix}. \tag{23}$$

That is, $\phi(\mathbf{d})(x)$ is an upper triangular Toepliz matrix, whose entry on the i-th superdiagonal is $d_i(x)$. The fact that $\mathbf{d} \in HS_m(K)$ is precisely the condition that $\phi(\mathbf{d})$ is a homomorphism. It is fairly easy to see that the two-sided vector space $V(\mathbf{d}) = {}_1K^{m+1}_{\phi(\mathbf{d})}$ is indecomposable in $\mathsf{Vect}(K)$. Conversely, if $\phi : K \to M_{m+1}(K)$ is a homomorphism such that $\phi(x)$ is an upper triangular Toepliz matrix for all $x \in K$, then $\mathbf{d} = \{d_0, \ldots, d_m\} \in HS_m(K)$, where $d_i(x)$ is the i-th superdiagonal of $\phi(x)$.

It follows readily that there is an abelian semigroup homomorphism $\Psi : \mathbb{Z}[HS(K)] \to K_0^{\oplus}(K)$. However, Ψ is in general neither injective nor surjective, and is also not a ring homomorphism.

For instance, let $\mathbf{d} = \{d_0, d_1\}$ and let $\mathbf{d}' = \{d_0, xd_1\}$ for some $x \in K^*$. Then conjugating $\phi(\mathbf{d})$ by $\operatorname{diag}(x, 1)$ shows that $V(\mathbf{d}) \cong V(\mathbf{d}')$ and so Ψ is not injective. Similarly, let $V = {}_1K^3_{\phi}$, where

$$\phi(x) = \begin{pmatrix} d_0(x) & d_1(x) & d_1(x) \\ 0 & d_0(x) & 0 \\ 0 & 0 & d_0(x) \end{pmatrix}.$$

Then V is indecomposable but is not in the image of Ψ, so Ψ is not surjective.

The fact that Ψ is not a ring homomorphism is easy: If $\mathbf{d} \in HS_m(K)$ and $\mathbf{d}' \in HS_n(K)$, then $\mathbf{d}{\cdot}\mathbf{d}' \in HS_{m+n}(K)$ and so the left dimension of $V(\mathbf{d}{\cdot}\mathbf{d}')$ is $m+n+1$. On the other hand, the left dimension of $V(\mathbf{d}) \otimes V(\mathbf{d}')$ is $(m+1)(n+1)$.

The above remarks show that in general $K_0^{\oplus}(K)$ is a more intractable object of study than $K_0^B(K)$, and that its structure depends on significantly subtler arithmetic properties of the field K.

5. Representatives for equivalence classes of matrix homomorphisms

Let $\phi : K \to M_n(K)$ be a homomorphism. In this final section, we consider the problem of finding a representative for the $\sim$-equivalence class of ϕ that has a particularly "nice" form.

For example, suppose that $\phi(y)$ has all of its eigenvalues in K for some $y \in K$. Then there exists $P \in GL_n(K)$ such that $P\phi(y)P^{-1}$ is in Jordan canonical form. Let $\lambda_1, \ldots, \lambda_t$ be the distinct eigenvalues of $\phi(y)$, with corresponding multiplicities $m_1, \ldots, m_t$. For each i, let $n_{i,1}, \ldots, n_{i,s_i}$ be the sizes of the λ_i-Jordan blocks of $P\phi(y)P^{-1}$. Then [2, Section VIII.2] implies the following result. (We say that an $m \times n$ matrix A is *generalized upper triangular Toepliz* if it is of the form $\begin{pmatrix} 0 & T \end{pmatrix}$ or $\begin{pmatrix} T \\ 0 \end{pmatrix}$, where T is an upper triangular Toepliz matrix.)

Theorem 5.1. *Assume the above notation. For all $x \in K$,*

$$P\phi(x)P^{-1} = \operatorname{diag}(\phi_1(x), \ldots, \phi_t(x)), \tag{24}$$

where each $\phi_i(x)$ is an $m_i \times m_i$-block matrix of the form $\phi_i(x) = (T_{ipq}(x))$, where $T_{ipq}(x)$ is a generalized upper triangular Toepliz matrix of size $n_{i,p} \times n_{i,q}$.

The above theorem uses nothing more than the description of the set of all matrices which commute with a given matrix in Jordan canonical form; in particular it does *not* use the additional information that ϕ is a homomorphism, or that the matrices in $\operatorname{im}\phi$ also commute with each other. Consequently one can often find a better representation than the one afforded by Theorem 5.1.

Example 5.2. Suppose that $\phi : K \to M_3(K)$ is such that $\phi(y) = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$ for some $y \in K$. Then $\phi(y)$ is in Jordan canonical form, so Theorem 5.1 shows that there exist functions $a, b, c, d, e : K \to K$ such that

$$\phi(x) = \begin{pmatrix} a(x) & b(x) & c(x) \\ 0 & a(x) & 0 \\ 0 & d(x) & e(x) \end{pmatrix}. \tag{25}$$

Writing out the condition that ϕ is a homomorphism shows that each of a and e are (nonzero) homomorphisms from K to K. If we conjugate $\phi(x)$ by the matrix $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$, we see that $\phi(x) \sim \psi(x) = \begin{pmatrix} a(x) & c(x) & b(x) \\ 0 & e(x) & d(x) \\ 0 & 0 & a(x) \end{pmatrix}$. If we let $V = {}_1K^3_\psi$, then the composition factors of V are $\{{}_1K_a, {}_1K_e, {}_1K_a\}$.

Suppose first that $a = e$. Then the fact that ψ is a homomorphism implies that $b(x_1x_2) = a(x_1)b(x_2) + c(x_1)d(x_2) + b(x_1)a(x_2)$ for all $x_1, x_2 \in K$. Since $b(x_1x_2) = b(x_2x_1)$ we can equate terms and get that $c(x_1)d(x_2) = c(x_2)d(x_1)$. If $c \neq 0$, then choosing x_2 so that $c(x_2) \neq 0$, we see that $d(x) = \alpha c(x)$, where $\alpha = d(x_2)/c(x_2)$. If $\alpha \neq 0$, then we can conjugate ψ by $Q = \operatorname{diag}(1, 1, \alpha)$ to conclude that $\phi \sim \begin{pmatrix} a & c & \frac{1}{\alpha}b \\ 0 & a & c \\ 0 & 0 & a \end{pmatrix}$. If $\alpha = 0$, then $d = 0$ and so $\phi \sim \begin{pmatrix} a & c & b \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}$. Finally, if $a \neq e$ then the fact that there are no nontrivial extensions between nonisomorphic simples shows that $\phi \sim \begin{pmatrix} a & b & 0 \\ 0 & a & 0 \\ 0 & 0 & e \end{pmatrix}$. Thus we conclude that ϕ is equivalent to a homomorphism as in Theorem 5.1 that is also upper triangular. $\square$

Motivated by the above example, we may ask whether a homomorphism ϕ is always equivalent to an upper triangular homomorphism or, ideally, an upper triangular homomorphism of the form (24). Assuming that the matrices in $\operatorname{im} \phi$ have their eigenvalues in K, the answer to the first question is “yes” [3, p. 100]. We shall prove that, under certain additional assumptions, the matrices in $\operatorname{im} \phi$ have upper triangular Toepliz diagonals. We then derive a sufficient condition for an affirmative answer to the second question. We begin with some elementary reductions.

Given $V \in \mathsf{Vect}(K)$, let $S_1, \ldots, S_t$ be a complete list of the pairwise nonisomorphic composition factors of V. Since $\operatorname{Ext}^1(S_i, S_j) = 0$ for $i \neq j$, we see that $V \cong V_1 \oplus \cdots \oplus V_t$, where each V_i has each of its composition factors isomorphic to S_i. Now, if ϕ represents V and ϕ_i represents V_i for each i, then it is clear that $\phi \sim \operatorname{diag}(\phi_1, \ldots, \phi_t)$. Thus it suffices to consider the case where the composition factors of ${}_1K^n_\phi$ are pairwise isomorphic. We shall further assume that the simple composition factor of ${}_1K^n_\phi$ is isomorphic to ${}_1K_a$ for some $a : K \to K$; we shall say that ϕ is *a-homogeneous* in this case.

Lemma 5.3. *If $\phi : K \to M_n(K)$ is a-homogenous for some $a : K \to K$, then ϕ is equivalent to an upper triangular homomorphism with each diagonal entry equal to a.*

Proof. We proceed by induction on n, the case $n = 1$ being trivial. Let $V = {}_1K^n_\phi$. Then ${}_1K_a$ is a sub-bimodule of V, generated as a left subspace by a single vector

v. Choose a basis for V containing v and order it so that v occurs last; then we see that, in this basis, $V \cong {}_1K^n_{\tilde{\phi}}$, where $\tilde{\phi} \sim \begin{pmatrix} \psi & \theta \\ 0 & a \end{pmatrix}$ for some $\psi : K \to M_{n-1}(K)$. Now, ${}_1K^{n-1}_{\psi}$ is also a-homogeneous and so by induction is equivalent to an upper triangular homomorphism with each diagonal entry equal to a. The result follows. □

Theorem 5.4. *Let $\phi : K \to M_n(K)$ be a-homogeneous for some $a : K \to K$. Then there exist higher derivations $\mathbf{d}_1, \ldots, \mathbf{d}_t$, each of whose 0-th components is equal to a, such that*

$$\phi \sim \begin{pmatrix} A_{11} & A_{12} & \ldots & A_{1t} \\ 0 & A_{22} & \ldots & A_{2t} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & A_{tt} \end{pmatrix} \tag{26}$$

where $A_{ii}(x) = \phi(\mathbf{d}_i)(x)$ and $A_{ij}(xy) = \sum_{l=1}^{t} A_{il}(x)A_{lj}(y)$ for all $x, y \in K$.

Proof. The fact that $A_{ij}(xy) = \sum_l A_{il}(x)A_{lj}(y)$ follows because ϕ is a homomorphism; the key is to show that the diagonal matrices A_{ii} have the stated form. By the previous lemma, we may assume without loss of generality that ϕ is upper triangular. Write $\phi = (\phi_{ij})$, where $\phi_{ii} = a$ for all i and $\phi_{ij} = 0$ for $i > j$. Let $i_1 \leq \cdots \leq i_q$ be all of the indices for which $\phi_{i_k, i_k+1} = 0$. Then we can partition ϕ into blocks of size $i_1, i_2 - i_1, \ldots, i_q - i_{q-1}, n - i_q$. If we let ϕ_l denote the l-th diagonal block in this partition, then each ϕ_l has the properties that each of its diagonal entries is equal to a, and none of its first superdiagonal entries is identically 0.

Replacing ϕ by ϕ_l we may assume without loss of generality that $\phi_{i,i+1}$ is not identically 0 for any i. After these reductions, we see that the theorem is trivially true when $n = 1$ or 2, so we assume without loss of generality that $n \geq 3$. If we expand out $\phi_{i,i+2}(xy)$ using the fact that ϕ is a homomorphism and $\phi_{ij} = 0$ for $i > j$, we obtain

$$\phi_{i,i+2}(xy) = \phi_{ii}(x)\phi_{i,i+2}(y) + \phi_{i,i+1}(x)\phi_{i+1,i+2}(y) + \phi_{i,i+2}(x)\phi_{i+2,i+2}(y) \tag{27}$$

and a similar equation for $\phi_{i,i+2}(yx)$. Substituting $\phi_{ii} = \phi_{i+2,i+2}$ and using the fact that $\phi(xy) = \phi(yx)$, we can simplify the resulting equations to obtain

$$\phi_{i,i+1}(x)\phi_{i+1,i+2}(y) = \phi_{i,i+1}(y)\phi_{i+1,i+2}(x) \tag{28}$$

for all $x, y \in K$. If we choose y such that $\phi_{i,i+1}(y) \neq 0$, then we have that $\phi_{i+1,i+2}(x) = \alpha_i \phi_{i,i+1}(x)$ for all $x \in K$, where $\alpha_i = \phi_{i+1,i+2}(y)/\phi_{i,i+1}(y)$. Note also that $\alpha_i \neq 0$ for any i since we know that $\phi_{i+1,i+2}$ is not identically 0.

Let $b = \phi_{12}$ and let $\beta_i = \prod_{j \leq i} \alpha_j$, so that

$$\phi = \begin{pmatrix} a & b & & & * \\ 0 & a & \beta_1 b & & \\ \vdots & 0 & a & \ddots & \\ \vdots & & \ddots & \ddots & \beta_{n-2} b \\ 0 & \ldots & \ldots & 0 & a \end{pmatrix}.$$

Choose $y \in K$ with $b(y) \neq 0$. An elementary calculation shows that $(\phi(y) - a(y)I_n)^{n-1}$ is the matrix whose only nonzero entry is $\beta_1 \ldots \beta_{n-2} b(y)^{n-1}$ in its $(1, n)$-position. This shows that the minimal polynomial for $\phi(y)$ is $(X - a(y))^n$, so that the Jordan canonical form for $\phi(y)$ is a single block of size n. If $P \in GL_n(K)$ is such that $P\phi(y)P^{-1}$ is in Jordan canonical form, then Theorem 5.1 shows that $P\phi(x)P^{-1}$ is an upper triangular Toepliz matrix with diagonal equal to $a(x)$ for all $x \in K$. Thus there exists a higher derivation $\mathbf{d}$ such that $P\phi P^{-1} = \phi(\mathbf{d})$. This shows that ϕ is equivalent to a matrix of the form (26). □

One may ask under what circumstances it is possible to obtain the best of both worlds: That is, when can we conclude that ϕ is equivalent to an upper triangular representation as in (26), and also have each A_{ij} be a generalized upper triangular Toepliz matrix as in Theorem 5.1? Since the Toepliz condition arises out of commuting with a matrix in Jordan canonical form, the following would be a sufficient condition:

($*$) Given a homomorphism ϕ, there exists $y \in K$ and $P \in GL_n(K)$ such that $P\phi(y)P^{-1}$ is in Jordan canonical form and $P\phi(x)P^{-1}$ is upper triangular for all $x \in K$.

If ϕ is an upper triangular homomorphism, then of course condition ($*$) is satisfied if there exists $y \in K$ and an upper triangular $P \in GL_n(K)$ such that $P\phi(y)P^{-1}$ is in Jordan canonical form.

Condition ($*$) is not automatic for a given y and ϕ. The following example illustrates that, given y, there may be no P such that $P\phi(y)P^{-1}$ is in Jordan canonical form and $P\phi(x)P^{-1}$ is upper triangular.

Example 5.5. Let $\mathbf{d} = \{d_0, d_1, d_2\}$ be a 2-derivation, and assume that $d_1 \neq 0$ and that there exists a $y \in K$ such that $d_1(y) = 0$, $d_2(y) \neq 0$. Define $\phi : K \to M_3(K)$ by

$$\phi(x) = \begin{pmatrix} d_0(x) & d_2(x) & d_1(x) \\ 0 & d_0(x) & 0 \\ 0 & d_1(x) & d_0(x) \end{pmatrix}.$$

We claim there does not exist a basis in which $\phi(y)$ has Jordan canonical form and the image of ϕ is upper triangular. To establish this claim, we describe every $P \in GL_3(K)$ in which the image of $P\phi P^{-1}$ is upper triangular, and show that $P\phi(y)P^{-1}$ is not in Jordan canonical form for any such P.

Since $d_1 \neq 0$, it is not hard to see that the only simultaneous eigenvectors for $\operatorname{im}\phi$ are in $W = \operatorname{span}(0, 1, 0)$. Similarly, the only simultaneous eigenvectors for $\operatorname{im}\phi$ acting on K^3/W are in $\operatorname{span}\{(0, 0, 1) + W\}$. From this we conclude that, if $\mathcal{B}$ is a basis with $\operatorname{im} P\phi P^{-1}$ upper triangular, then

$$\mathcal{B} = \{(0, f_1, 0), (0, f_2, f_3), (f_4, f_5, f_6) : f_1, f_3, f_4 \neq 0\}.$$

For such a basis $\mathcal{B}$, we have

$$P\phi(x)P^{-1} = \begin{pmatrix} d_0(x) & \frac{f_4}{f_3}d_1(x) & (\frac{f_6f_3-f_4f_2}{f_1f_3})d_1(x) + \frac{f_4}{f_1}d_2(x) \\ 0 & d_0(x) & \frac{f_3}{f_1}d_1(x) \\ 0 & 0 & d_0(x) \end{pmatrix}.$$

By construction, $P\phi(y)P^{-1}$ is not in Jordan canonical form.

Note that higher derivations satisfying the given hypotheses do exist. For example, let K be the quotient field of $k[x, y, z]/(xy - z^2)$, where k is a field of characteristic 2. In [7, Example 1.2 and Theorem 1.5], a nontrivial $\mathbf{d} \in HS_2(K)$ is constructed such that $d_1(x - z) = 0$ and $d_2(x - z) = x$. □

Definition 5.6. Let A be an $n \times n$ upper triangular matrix with single eigenvalue λ, and let the Jordan canonical form of A have block sizes $n_1 \geq n_2 \geq \cdots \geq n_p$. For each $i \leq n$, let A_i be the matrix consisting of the first i rows and columns of A. We say that A is *Jordan-ordered* if, for all $i \leq n$, the dimension of the eigenspace of A_i is j, where j is the smallest integer such that $n_1 + \cdots + n_j \geq i$.

Example 5.7. Let $A = \begin{pmatrix} \lambda & 0 & 1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$, so that the Jordan canonical form of A has blocks of size 2 and 1. Then A is not Jordan-ordered, because the dimension of the eigenspace of $A_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ is 2 and not 1. □

It is not hard to see that, if A is in Jordan canonical form, then A is Jordan-ordered if and only if the Jordan blocks of A are arranged in decreasing size.

The following is our main result concerning Jordan-ordered matrices.

Theorem 5.8. *If $A \in M_n(K)$ is Jordan-ordered, then there exists an upper triangular $P \in GL_n(K)$ such that PAP^{-1} is Jordan-ordered and is in Jordan canonical form.*

We begin with a preliminary lemma.

Lemma 5.9. *Suppose that $A \in M_{n+1}(K)$ has a single eigenvalue λ of multiplicity n. If*

$$A = \begin{pmatrix} & & & a_1 \\ & B & & \vdots \\ & & & a_n \\ 0 & \dots & 0 & \lambda \end{pmatrix} \tag{29}$$

with $B \in M_n(K)$ in Jordan canonical form, then there exists an upper triangular $P \in GL_n(K)$ such that PAP^{-1} is in Jordan canonical form.

Proof. Since A has the single eigenvalue λ, the Jordan canonical form for A must be

$$\begin{pmatrix} & & & & 0 \\ & B & & & \vdots \\ & & & & 0 \\ & & & & a \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix}, \tag{30}$$

with $a = 0$ or 1. We give the proof when $a = 1$, the case $a = 0$ being similar and left to the reader.

Let E_A denote the eigenspace of A and suppose that $\dim E_A = m + 1$. Since $e_{n+1} \in E_A$, we can take a basis for E_A containing it; moreover elementary calculations then allow us to assume that the final entry of all other basis elements is 0. Thus E_A has a basis of the form

$$\{(0, \dots, 0, 1), (c_{11}, \dots, c_{1n}, 0), \dots, (c_{m1}, \dots, c_{mn}, 0)\}.$$

Since the last m of these vectors are eigenvectors for A, we see that $(a_1, \dots, a_n)$ must be a solution to the system of equations

$$\begin{aligned} c_{11}x_1 + \cdots + c_{1n}x_n &= 0 \\ &\vdots \\ c_{m1}x_1 + \cdots + c_{mn}x_n &= 0 \end{aligned} \tag{31}$$

and that

$$\{(c_{11}, \dots, c_{1n}), \dots, (c_{m1}, \dots, c_{mn})\}$$

is a set of m linearly independent eigenvectors of B. Because $a = 1$ we see that the dimension of the eigenspace E_B of B is also $m+1$, and we note that $e_n \in E_B$. Since B is in Jordan canonical form, the matrix

$$\begin{pmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{m1} & \dots & c_{mn} \end{pmatrix} \tag{32}$$

has $n - m - 1$ of its columns equal to 0, and its final column cannot be equal to 0 since e_n is an eigenvector for B. Thus (31) can be viewed as a system of m equations in $m+1$ variables, say $x_{i_1}, \dots, x_{i_{m+1}} = x_n$. Since the rows of (32) are linearly independent, some subset of m columns of (32) is linearly independent. Thus the solution space of (31) is 1-dimensional. On the other hand, since A has the given Jordan canonical form, $(x_{i_1}, \dots, x_{i_{m+1}}) = (0, 0, \dots, 1)$ must be a solution to (31). Thus we conclude that $(a_{i_1}, \dots, a_{i_{m+1}}) = (0, 0, \dots, c)$ for some $c \in K$.

Consider the system

$$(\lambda I_n - B)\begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} a_1 \\ \vdots \\ a_{n-1} \\ 0 \end{pmatrix}.$$

Since B is in Jordan canonical form, the image of left multiplication by $\lambda I_n - B$ has each of its $i_1, \dots, i_{m+1}$-components equal to 0, and also has dimension $n - m - 1$. Since $a_{i_1} = \cdots = a_{i_m} = 0$, we see that there is a solution $y_1 = b_1, \dots, y_n = b_n$. Let $\vec{b}$ be the column vector $(b_1, \dots, b_n)^T$; then an elementary calculation shows that, if $P = \begin{pmatrix} I_n & \vec{b} \\ 0 & 1 \end{pmatrix} \in GL_{n+1}(K)$, then

$$PAP^{-1} = \begin{pmatrix} & & & & 0 \\ & B & & & \vdots \\ & & & & 0 \\ & & & & c \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix}.$$

It follows, since the Jordan canonical form for A is (30), that $c \neq 0$. Conjugating by $\operatorname{diag}(1, \dots, 1, 1/c)$ finishes the proof. □

Proof of Theorem 5.8. We proceed by induction on n, the case $n = 1$ being trivial. Since A is upper triangular, e_n is an eigenvector for A. If A_{n-1} denotes the matrix obtained by deleting the last row and column from A, then by induction there exists an upper triangular $Q \in GL_{n-1}(K)$ such that $QA_{n-1}Q^{-1}$ is Jordan-ordered and in Jordan canonical form. Let $R = \begin{pmatrix} Q & 0 \\ 0 & 1 \end{pmatrix} \in GL_n(K)$; conjugating A by R then gives

$$RAR^{-1} = \begin{pmatrix} & & & a_1 \\ & B & & \vdots \\ & & & a_n \\ 0 & \dots & 0 & \lambda \end{pmatrix},$$

where B is the Jordan ordered, Jordan canonical form for A_{n-1}.

Let the Jordan canonical form for A have blocks of sizes $n_1 \geq \cdots \geq n_p$. If $n_p = 1$, then B has blocks of sizes $n_1, \dots, n_{p-1}$, and the Jordan canonical form for A is $\begin{pmatrix} B & 0 \\ 0 & \lambda \end{pmatrix}$. By Lemma 5.9, there is an upper triangular $T \in GL_n(K)$ with $TRAR^{-1}T^{-1}$ Jordan-ordered and in Jordan canonical form. Thus the theorem follows with $P = TR$ in this case.

If $n_p > 1$, then B has blocks of sizes $n_1, \dots, n_{p-1}, n_p - 1$ and the block of size $n_p - 1$ occurs at the bottom of B. Thus the Jordan canonical form for A is

$$\begin{pmatrix} & & & & 0 \\ & B & & & \vdots \\ & & & & 0 \\ & & & & 1 \\ 0 & \dots & \dots & 0 & \lambda \end{pmatrix},$$

and again letting T be as in Lemma 5.9, we see that PAP^{-1} is Jordan-ordered and in Jordan canonical form for $P = TR$. □

Combining Theorems 5.4 and 5.8, we can state a sufficient condition for a homomorphism $\phi : K \to M_n(K)$ to be equivalent to an upper triangular homomorphism which is generalized upper triangular Toepliz. We state the result in the case where ϕ is a-homogenous for some $a : K \to K$.

Corollary 5.10. *Let ϕ be a-homogeneous, and let $\psi \sim \phi$, where ψ is a homomorphism in the form* (26). *If $\psi(y)$ is Jordan-ordered for some $y \in K$, then*

$$\phi \sim \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1s} \\ 0 & T_{22} & \dots & T_{2s} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & T_{ss} \end{pmatrix} \tag{33}$$

where each $T_{ij}(x)$ is generalized upper triangular Toepliz.

In particular there exist higher derivations $\mathbf{d}_1, \dots, \mathbf{d}_s$ such that $T_{ii} = \phi(\mathbf{d}_i)$, although these derivations may be different than those in (26).

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