.

CORRIGENDUM TO "LOCAL DUALITY FOR CONNECTED $\mathbb{Z}$-ALGEBRAS" [J. PURE APPL. ALGEBRA, 225(9) (2019) PAPER NO. 106676]

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ABSTRACT. The statements of Theorem 6.11 and Theorem 7.8 in the published paper are not correct: each have a missing hypothesis which is needed for their proof. We restate both theorems (with the required extra hypothesis), and give corrected proofs of each.

1. Notation and Terminology

Throughout this note, k will denote a field over which all objects are defined. We adopt the same terminology and notation as in [3].

2. Replacement for Theorem 6.11

The proof of [3, Theorem 6.11] is not correct, as the expression $M \underline{\otimes}_A^{\mathrm{L}} \mathrm{R}\,\tau(A)$ which appears in the second paragraph of the proof of [3, Theorem 6.11], is undefined if $\mathrm{R}\,\tau(A)$ isn't bounded above. However, the proof is correct with the additional hypothesis that τ has finite cohomological dimension. This hypothesis is used in exactly the same way as it is used in the proof of [4, Theorem 5.1], as we describe below. In fact, with this additional hypothesis, a slightly more general version of [3, Theorem 6.11] holds:

Theorem 2.1. *Let A be a right Ext-finite connected $\mathbb{Z}$-algebra, and suppose τ has finite cohomological dimension as a functor on $\mathsf{Gr}A$. Let $M^\bullet$ be an object in $\mathsf{D}^-(\mathsf{Bimod}(K-A))$. Then*

$$D\,\mathrm{R}\,\tau(M^\bullet) \cong \mathrm{R}\,\underline{\mathcal{H}om}_A(M^\bullet, D\,\mathrm{R}\,\tau(A))$$

in $\mathsf{D}(\mathsf{Bimod}(A-K))$.

Proof. Since τ has finite cohomological dimension as a functor on $\mathsf{Gr}A$, it's extension to $\mathsf{Bimod}(K-A)$ also has finite cohomological dimension by [3, Lemma 5.9]. Thus, although $M^\bullet$ is in $\mathsf{D}^-(\mathsf{Bimod}(K-A))$, $D\,\mathrm{R}\,\tau(M^\bullet)$ is defined by [1, p. 57].

Next, we let $E^\bullet$ denote an injective resolution of A in $\mathsf{Bimod}(A-A)$, which exists by [3, Proposition 2.2(1)]. Then, following the proof of [4, Theorem 5.1], since the cohomological dimension of τ is finite, we may take a standard truncation $F^\bullet$ of $E^\bullet$ in such a way that $F^\bullet$ is *bounded*, $F^\bullet$ consists of τ-acyclic terms, and $F^\bullet$ is quasi-isomorphic to $E^\bullet$. It follows that $D\,\mathrm{R}\,\tau(A) = D\tau(F^\bullet)$. Furthermore, if $L^\bullet \in \mathsf{D}^-(\mathsf{Bimod}(K-A))$ is a projective resolution of $M^\bullet$ of the form specified in the first paragraph of the proof of [3, Theorem 6.11], then $\mathrm{Tot}(L^\bullet \underline{\otimes}_A F^\bullet)$ is a bounded above complex of τ-acyclic terms by the proof of [3, Lemma 6.9]. Moreover, by [3, Proposition 6.8], $\mathrm{Tot}(L^\bullet \underline{\otimes}_A F^\bullet)$ is quasi-isomorphic to $M^\bullet$. Thus, by the hypothesis

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on τ and [1, p. 57], $\mathrm{R}\,\tau(M^\bullet)$ may be computed via the complex $\tau\,\mathrm{Tot}(L^\bullet \underline{\otimes}_A F^\bullet) \cong \mathrm{Tot}(L^\bullet \underline{\otimes}_A \tau(F^\bullet))$, where the last isomorphism is by [3, Lemma 6.10].

Thus, by [3, Corollary 6.2],

$$\begin{aligned} \mathrm{R}\,\underline{\mathcal{H}om}_A(M^\bullet, D\,\mathrm{R}\,\tau(A)) &\cong \underline{\mathcal{H}om}^\bullet_A(L^\bullet, D\tau(F^\bullet)) \\ &\cong D(\mathrm{Tot}(L^\bullet \underline{\otimes}_A \tau(F^\bullet))) \\ &\cong D\,\mathrm{R}\,\tau(M^\bullet). \end{aligned}$$

□

We let $\mathrm{cd}\,\tau$ denote the cohomological dimension of τ, and we define the small global dimension of A by $\mathrm{sgldim}\,A := \sup\{\mathrm{pd}\,e_iA_0 \mid i \in \mathbb{Z}\}$. We note that if A is a connected, Ext-finite $\mathbb{Z}$-algebra which is either AS-regular ([3, Definition 7.1]) or ASF-regular ([3, Definition 7.5]), then $\mathrm{sgldim}\,A < \infty$ by definition.

Lemma 2.2. *Suppose A is a connected, Ext-finite $\mathbb{Z}$-algebra such that* $\mathrm{sgldim}\,A < \infty$*. Then* $\mathrm{cd}\,\tau < \infty$*.*

Proof. In light of [3, Lemma 5.8], the proof is the same as the proof of the first assertion in [3, Proposition 7.3]. □

It follows from Lemma 2.2 that the proof of [3, Theorem 7.10], which relies on [3, Theorem 6.11], is correct as originally written.

3. Replacement for Theorem 7.8

The proof of [3, Theorem 7.8] is incomplete. The issue is that it is unclear whether the isomorphism given in the second paragraph of the proof is natural with respect to maps $e_iA \to e_jA$. In order for this to be the case, it would suffice for the isomorphism $\mathrm{R}^d\,\tau(e_iA) \cong D(Ae_{i-l})$ given in [3, Definition 7.5] to be suitably natural. Without an additional hypothesis on A (for example that there is an isomorphism of $\mathbb{Z}$-algebras $A \to A(-l)$, where $A(-l)$ is defined below), it is unclear what the correct notion of naturality would be. However, inspired by [2, Theorem 3.3], we replace [3, Theorem 7.8] with Theorem 3.7 below, in which the ASF-regular hypothesis is replaced by a stronger hypothesis. In the $\mathbb{Z}$-graded case, this stronger hypothesis holds for noetherian AS-regular k-algebras [2, Theorem 1.2]. Before stating the definition of this stronger regularity condition in Definition 3.1, we introduce some terminology.

If A is a $\mathbb{Z}$-algebra and $l \in \mathbb{Z}$, we let $A(l)$ denote the $\mathbb{Z}$-algebra with $A(l)_{ij} := A_{i+l,j+l}$ and with multiplication induced by that of A. If B is a $\mathbb{Z}$-algebra, M is an object of $\mathsf{Bimod}(A-B)$ and $r, l \in \mathbb{Z}$, we let $M(r,l)$ denote the object of $\mathsf{Bimod}(A(r)-B(l))$ with $M(r,l)_{ij} := M_{i+r,j+l}$ and with multiplication induced by that of A and B on M. If C is a $\mathbb{Z}$-algebra and $\phi : C \to B$ is a morphism, we define M_ϕ in $\mathsf{Bimod}(A-C)$ as the module with the same underlying set and left action but with right action defined by $m * c := m\phi(c)$. Finally, if $\psi : C \to A$ is a morphism of $\mathbb{Z}$-algebras, we define ${}_\psi M$ in $\mathsf{Bimod}(C-B)$ similarly.

Definition 3.1. A connected $\mathbb{Z}$-algebra A is called *regular of dimension d and of Gorenstein parameter l* if

(1) $\mathrm{sgldim}\,A = d < \infty$, and

(2) $D\,\mathrm{R}\,\tau(A) \cong A(0,-l)_\nu[d]$ in $\mathsf{D}(\mathsf{Bimod}(A-A))$ for some isomorphism
$$\nu : A \to A(-l)$$
of $\mathbb{Z}$-algebras called the *Nakayama isomorphism*.

It is straightforward to show that if A is regular, then it is ASF-regular.

For the remainder of this section, we build towards the replacement of [3, Theorem 7.8]. We leave the straightforward details of the next result to the reader.

Lemma 3.2. *Let $\Phi : \mathsf{Bimod}(A-B) \to \mathsf{Bimod}(\widetilde{B^{op}}-\widetilde{A^{op}})$ denote the functor defined on objects by $\Phi(M)_{ij} := M_{-j,-i}$ and with bimodule action induced by that of A and B, and on morphisms by $\Phi(f)_{ij} = f_{-j,-i}$. Then Φ is an equivalence (in fact, an isomorphism) of categories.*

Remark 3.3. In the sequel, we will sometimes utilize the equivalence Φ of Lemma 3.2, as well as the equivalences from [3, Proposition 2.2], implicitly.

Suppose A is a $\mathbb{Z}$-algebra, $l \in \mathbb{Z}$, and $\nu : A \to A(-l)$ is an isomorphism. Let $\tilde{\nu} : \widetilde{A^{op}} \to \widetilde{A^{op}}(l)$ denote the isomorphism defined by $\tilde{\nu}_{ij} = \nu_{-j,-i}$. Abusing notation slightly, $\tilde{\nu}^{-1} : \widetilde{A^{op}} \to \widetilde{A^{op}}(-l)$.

Lemma 3.4. *With the notation above, $\Phi(A(0,-l)_\nu) \cong \widetilde{A^{op}}(0,-l)_{\tilde{\nu}^{-1}}$.*

Proof. First, we claim that $\Phi(A(0,-l)_\nu) = {}_{\tilde{\nu}}\widetilde{A^{op}}(l,0)$. To prove the claim, we compute
$$\Phi(A(0,-l)_\nu)_{ij} = A(0,-l)_{-j,-i} = A_{-j,-i-l} = \widetilde{A^{op}}(l,0)_{ij}.$$
Next, suppose $x \in (\widetilde{A^{op}})_{jk} = A_{-k,-j}$. Then we have
$$\Phi(A(0,-l)_\nu)_{ij} \cdot x = xA_{-j,-i-l} = \widetilde{A^{op}}_{i+l,j} \cdot x,$$
while
$$x \cdot \Phi(A(0,-l)_\nu)_{kq} = (A(0,-l)_\nu)_{-q,-k} * x = A_{-q,-l-k}\nu(x) = \tilde{\nu}(x)\widetilde{A^{op}}(l,0)_{kq},$$
proving the claim.

To complete the proof, we note that $\tilde{\nu}^{-1} : {}_{\tilde{\nu}}\widetilde{A^{op}}(l,0) \to \widetilde{A^{op}}(0,-l)_{\tilde{\nu}^{-1}}$ defines an isomorphism of $\widetilde{A^{op}} - \widetilde{A^{op}}$-bimodules, as one can check. $\square$

We define a functor $I_0 : \mathsf{Gr}A \to \mathsf{Bimod}(K-A)$ on objects by letting
$$I_0(M)_{ij} = \begin{cases} M_j & \text{if } i = 0 \\ 0 & \text{otherwise,} \end{cases}$$
and on morphisms in the obvious way. We define functors $J_0 : A-\mathsf{Gr} \to \mathsf{Bimod}(A-K)$ and $\tilde{I}_0 : \mathsf{Gr}\widetilde{A^{op}} \to \mathsf{Bimod}(K-\widetilde{A^{op}})$ similarly.

Remark 3.5. Suppose P is an object of $\mathsf{Bimod}(A-K)$ such that $P_{ij} = 0$ for $j \neq 0$. Then $\tilde{I}_0(Pe_0) = P$. A similar observation applies to morphisms between such modules.

Lemma 3.6. *Let A be a $\mathbb{Z}$-algebra, let M denote an object of $\mathsf{Gr}A$ and let N denote an object of $\mathsf{Bimod}(A-A)$. Then*

(1) *evaluation induces a morphism of right A-modules*
$$\Gamma : I_0(M) \to \underline{\mathcal{H}om}_{\widetilde{A^{op}}}(\underline{\mathcal{H}om}_A(I_0(M), N), N) \tag{3-1}$$
which is natural in M, and compatible with finite direct sums, and

(2) *if A is connected and locally finite, i.e. A_{ij} is finite-dimensional over A_{ii} and A_{jj} for all $i, j \in \mathbb{Z}$, $M = e_i A$ and $N = A(0, -l)_\nu$ where $\nu : A \to A(-l)$ is an isomorphism, then (3-1) is an isomorphism.*

Proof. We begin by proving the existence of (3-1), but note that some care is required, as we identify $\underline{\mathcal{H}om}_A(I_0(M), N)$, an object in $\mathsf{Bimod}(A - K)$, with its associated object in $\mathsf{Bimod}(K - \widetilde{A^{op}})$ via the equivalence given by Lemma 3.2. A similar comment, applied to N, holds. Thus, considering $\underline{\mathcal{H}om}_{\widetilde{A^{op}}}(\underline{\mathcal{H}om}_A(I_0(M), N), N)$ as an object in $\mathsf{Bimod}(K - A)$, we have

$$\begin{aligned} \underline{\mathcal{H}om}_{\widetilde{A^{op}}}(\underline{\mathcal{H}om}_A(I_0(M), N), N)_{0,i} &= \mathcal{H}om_{\widetilde{A^{op}}}(\underline{\mathcal{H}om}_A(e_0 I_0(M), N), \tilde{e}_{-i}\Phi(N)) \\ &= \mathcal{H}om_{\widetilde{A^{op}}}(\underline{\mathcal{H}om}_A(e_0 I_0(M), N), \oplus_j N_{-j,i}). \end{aligned}$$

We define

$$\Gamma_i : I_0(M)_{0,i} \to \mathcal{H}om_{\widetilde{A^{op}}}(\underline{\mathcal{H}om}_A(e_0 I_0(M), N), \oplus_j N_{-j,i})$$

by sending $m \in M_i = I_0(M)_{0,i}$ to the evaluation map

$$\mathrm{ev}_m : \underline{\mathcal{H}om}_A(e_0 I_0(M), N) \to \oplus_j N_{-j,i}.$$

To show that this map is well defined, we note that if $\psi \in \Phi(\underline{\mathcal{H}om}_A(e_0 I_0(M), N))$, then $\psi_l \in \mathcal{H}om_A(e_0 I_0(M), e_{-l} N)$ so that $\psi_l(m) \in N_{-li} \subset \oplus_j N_{-j,i}$. Thus, to conclude the proof that Γ_i is well-defined, we must show that ev_m respects right multiplication by elements of $\widetilde{A^{op}}$. To this end, let $\tilde{a} \in (\widetilde{A^{op}})_{lk}$, write $a \in A_{-k,-l}$ for the corresponding element of A, and let $\psi \in \Phi(\underline{\mathcal{H}om}_A(e_0 I_0(M), N))$. Then

$$\mathrm{ev}_m(\psi_l \cdot \tilde{a}) = (\psi_l \cdot \tilde{a})(m) = (a \cdot \psi_l)(m) = a \cdot \psi_l(m) = \mathrm{ev}_m(\psi_l) \cdot \tilde{a}.$$

It follows that Γ_i is a well-defined function.

Next, we show Γ is a map of right A-modules. Let $a_{il} \in A_{il}$, write $\tilde{a}_{-l,-i}$ for the corresponding element of $(\widetilde{A^{op}})_{-l,-i}$, and let $m \in M_i = I_0(M)_{0,i}$. Then $\Gamma(m \cdot a_{il}) = \mathrm{ev}_{m \cdot a_{il}}$. On the other hand,

$$\mathrm{ev}_m \cdot a_{il}(\psi) = (\tilde{a}_{-l,-i} \cdot \mathrm{ev}_m)(\psi) = \tilde{a}_{-l,-i} \cdot \psi(m) = \psi(m) \cdot a_{i,l}.$$

Since $\psi \in \underline{\mathcal{H}om}_A(e_0 I_0(M), N)$, these expressions are equal. The fact that Γ is compatible with addition is trivial. Finally, the fact that Γ is natural in M and compatible with finite direct sums is straightforward and left to the reader.

We next prove that, in case $M = e_i A$ and $N = A(0, -l)_\nu$, then Γ is an isomorphism. To prove injectivity, let $a_{ik} \in (e_i A)_k$ be nonzero. We note that $\nu \in \mathcal{H}om_A(e_i A, e_{i-l} A(0, -l)_\nu)$, and $\mathrm{ev}_{a_{ik}}(\nu) = \nu(a_{ik}) \neq 0$. It follows that Γ is injective.

To prove Γ is surjective, it suffices, by the locally finite hypothesis, to show that $\underline{\mathcal{H}om}_{\widetilde{A^{op}}}(\underline{\mathcal{H}om}_A(I_0(e_i A), A(0, -l)_\nu), A(0, -l)_\nu)_{0,j} \cong A_{ij}$ as right A_{jj}-modules. This follows from the isomorphism below (and its analogous version for $\widetilde{A^{op}}$):

$$\underline{\mathcal{H}om}_A(e_0 I_0(e_i A), A(0, -l)_\nu) \cong J_0(A e_{i-l}) = \tilde{I}_0(\tilde{e}_{-i+l} \widetilde{A^{op}})$$

which we leave as an exercise to the reader. $\square$

Theorem 3.7 below should replace [3, Theorem 7.8]. To state it, we introduce some terminology. We let $\mathsf{D}_c(\mathsf{Gr}A)$ denote the full subcategory of $\mathsf{D}(\mathsf{Gr}A)$ consisting of complexes with coherent cohomology. Let $\mathsf{D}_c^b(\mathsf{Gr}A)$, $\mathsf{D}_c^+(\mathsf{Gr}A)$, and $\mathsf{D}_c^-(\mathsf{Gr}A)$ denote the intersection of $\mathsf{D}_c(\mathsf{Gr}A)$ with $\mathsf{D}^b(\mathsf{Gr}A)$, $\mathsf{D}^+(\mathsf{Gr}A)$, and $\mathsf{D}^-(\mathsf{Gr}A)$, respectively.

Theorem 3.7. *Suppose A is a connected, (left- and right-) coherent $\mathbb{Z}$-algebra. Suppose, further, that both A and $\widetilde{A^{op}}$ are regular of dimension d and of Gorenstein parameter l with Nakayama automorphisms ν and $\tilde{\nu}^{-1}$, respectively. Then*

$$(D\,\mathrm{R}\,\tau(I_0(-)))e_0 : \mathsf{D}^b_c(\mathsf{Gr}A) \leftrightarrow \mathsf{D}^b_c(\mathsf{Gr}\widetilde{A^{op}}) : (D\,\mathrm{R}\,\tilde{\tau}(\tilde{I}_0(-)))\tilde{e}_0$$

is a duality.

Proof. We begin by noticing that since A is coherent and connected, it is Ext-finite.

We now claim that $(D\,\mathrm{R}\,\tau(I_0(-)))e_0 : \mathsf{D}(\mathsf{Gr}A)^{op} \to \mathsf{D}(A-\mathsf{Gr})$ is way-out in both directions. Since

$$(-)e_0 : \mathsf{Bimod}(A-K) \to A-\mathsf{Gr}$$

is exact, and D is exact, it suffices to show

$$\mathrm{R}\,\tau(I_0(-)) : \mathsf{D}(\mathsf{Gr}A) \to \mathsf{D}(\mathsf{Bimod}(K-A))$$

is way-out in both directions. To prove this assertion, we note that since $\tau : \mathsf{Gr}A \to \mathsf{Gr}A$ has finite cohomological dimension by Lemma 2.2, [3, Lemma 6.9] implies that the same holds for $\tau(I_0(-))$, so that the claim follows from [1, Example 1, p. 68].

Next, we show that if $M^\bullet$ is an object in $\mathsf{D}^b(\mathsf{Gr}A)$, then $(D\,\mathrm{R}\,\tau(I_0(M^\bullet)))e_0$ is an object in $\mathsf{D}^b(A-\mathsf{Gr})$. To prove this, it suffices to show $\mathrm{R}\,\tau(I_0(M^\bullet))$ is bounded. However, as above, we note that $\tau(I_0(-))$ has finite cohomological dimension, and so the assertion follows as in the second paragraph of the proof of Theorem 2.1.

We now claim that $(D\,\mathrm{R}\,\tau(I_0(-)))e_0$ induces a functor from $\mathsf{D}^-_c(\mathsf{Gr}A)^{op}$ to the category $\mathsf{D}_c(A-\mathsf{Gr})$. To prove this, we first note that $\mathsf{coh}A$ is a thick subcategory of $\mathsf{Gr}A$, so that $\mathsf{D}_c(\mathsf{Gr}A)$ and $\mathsf{D}_c(A-\mathsf{Gr})$ are well-defined full subcategories of $\mathsf{D}(\mathsf{Gr}A)$ and $\mathsf{D}(A-\mathsf{Gr})$, respectively. Thus, we consider

$$F := (D\,\mathrm{R}\,\tau(I_0(-)))e_0 : \mathsf{D}^-_c(\mathsf{Gr}A)^{op} \to \mathsf{D}(A-\mathsf{Gr})$$

and show that this factors through $\mathsf{D}_c(A-\mathsf{Gr})$. By [1, Proposition 7.3(iii), Chapter 1], it suffices to show that $F(X)$ is an object of $\mathsf{D}_c(A-\mathsf{Gr})$ for all coherent X. By the reversed form of [1, Proposition 7.3(iv), Chapter 1], it suffices to show that for every free and finitely generated A-module E, $F(E)$ is an object of $\mathsf{D}_c(A-\mathsf{Gr})$. But

$$\begin{aligned} F(E) = (D\,\mathrm{R}\,\tau(I_0(E)))e_0 &\cong \mathrm{R}\,\underline{\mathcal{H}om}_A(I_0(E), D\,\mathrm{R}\,\tau(A))e_0 \\ &\cong \underline{\mathcal{H}om}_A(E, A(0,-l)_\nu)[d] \end{aligned}$$

where the first isomorphism is due to Theorem 2.1, and the second follows from the definition of regularity. Since the last module is free and finitely generated as the reader can check, it is coherent, and so the claim follows.

Finally, we show that if $M^\bullet$ is an object of $\mathsf{D}^b_c(\mathsf{Gr}A)$, then there is a natural isomorphism

$$M^\bullet \longrightarrow (D\,\mathrm{R}\,\tilde{\tau}(\tilde{I}_0(D\,\mathrm{R}\,\tau(I_0(M^\bullet)))e_0))\tilde{e}_0$$

To prove this, we note that

$$\begin{aligned} D\,\mathrm{R}\,\tilde{\tau}(\tilde{I}_0(D\,\mathrm{R}\,\tau(I_0(M^\bullet))e_0)) &\cong D\,\mathrm{R}\,\tilde{\tau}(D\,\mathrm{R}\,\tau(I_0(M^\bullet))) \\ &\cong \mathrm{R}\,\underline{\mathcal{H}om}_{\widetilde{A^{op}}}(D\,\mathrm{R}\,\tau(I_0(M^\bullet)), \widetilde{A^{op}}(0,-l)_{\tilde{\nu}^{-1}})[d] \end{aligned}$$

where the first isomorphism follows from Remark 3.5, which applies in light of the fact that $(D\,\mathrm{R}\,\tau(I_0(M^\bullet)))e_j = D(e_j\,\mathrm{R}\,\tau(I_0(M^\bullet))) = D\,\mathrm{R}\,\tau(e_jI_0(M^\bullet)) = 0$ for every $j \neq 0$, while the second isomorphism follows from Theorem 2.1 and the fact that

$\widetilde{A^{op}}$ is regular. By another application of Theorem 2.1, we find the last expression is isomorphic to

$$\mathrm{R}\,\underline{\mathcal{H}om}_{\widetilde{A^{op}}}(\mathrm{R}\,\underline{\mathcal{H}om}_A(I_0(M^\bullet), A(0,-l)_\nu), \widetilde{A^{op}}(0,-l)_{\tilde{\nu}^{-1}}). \tag{3-2}$$

Since, by Lemma 3.4, $A(0,-l)_\nu \cong \widetilde{A^{op}}(0,-l)_{\tilde{\nu}^{-1}}$ as bimodules, the argument in [5, p. 52] can be adapted to our setting (using Lemma 3.6(1)) to show that there is a natural map from $I_0(M^\bullet)$ to (3-2), induced by evaluation. We need to show that if $M^\bullet$ is an object of $\mathsf{D}^b_c(\mathsf{Gr}A)$, then this map is an isomorphism. By [1, Proposition 7.1(i), Chapter 1], it suffices to show that this is an isomorphism when $M^\bullet$ is a coherent module M concentrated in degree zero. By the reversed form of [1, Proposition 7.1(iv), Chapter 1], it suffices to prove this when M is a finitely generated free module. This follows from Lemma 3.6(2), since in this case, $\underline{\mathcal{H}om}_A(I_0(M), A(0,-l)_\nu)e_0$ is a free and finitely generated right $\widetilde{A^{op}}$-module. $\square$

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