GRASSMANNIANS OF TWO-SIDED VECTOR SPACES

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ABSTRACT. Let $k \subset K$ be an extension of fields, and let $A \subset M_n(K)$ be a k-algebra. We study parameter spaces of m-dimensional subspaces of K^n which are invariant under A. The space $\mathbb{F}_A(m,n)$, whose R-rational points are A-invariant, free rank m summands of R^n, is well known. We construct a distinct parameter space, $\mathbb{G}_A(m,n)$, which is a fiber product of a Grassmannian and the projectivization of a vector space. We then study the intersection $\mathbb{F}_A(m,n) \cap \mathbb{G}_A(m,n)$, which we denote by $\mathbb{H}_A(m,n)$. Under suitable hypotheses on A, we construct affine open subschemes of $\mathbb{F}_A(m,n)$ and $\mathbb{H}_A(m,n)$ which cover their K-rational points. We conclude by using $\mathbb{F}_A(m,n)$, $\mathbb{G}_A(m,n)$, and $\mathbb{H}_A(m,n)$ to construct parameter spaces of two-sided subspaces of two-sided vector spaces.

CONTENTS

1. INTRODUCTION

Throughout this paper, $k \subset K$ is an extension of fields. By a *two-sided vector space* we mean a k-central $K-K$-bimodule V which is finite-dimensional as a left K-module. Thus, a two-sided vector space on which K acts centrally is just a finite-dimensional vector space over K. The purpose of this paper is to continue the classification of two-sided vector spaces begun in [4] by constructing and studying parameter spaces of two-sided subspaces of V.

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Instead of focusing exclusively on parameter spaces of two-sided subspaces of V, we take a more general perspective. Let $A \subset M_n(K)$ be a k-algebra, and let R be a K-algebra. The functor $F_A(m,n) : K - \mathsf{alg} \to \mathsf{Sets}$ defined on objects by

$$F_A(m,n)(R) = \{\text{free rank } m \text{ direct summands of } R^n \text{ which are } A\text{-invariant}\}$$

and on morphisms by pullback is representable by a subscheme, $\mathbb{F}_A(m,n)$, of the Grassmannian of m-dimensional subspaces of K^n, $\mathbb{G}(m,n)$ [3]. The scheme $\mathbb{F}_A(m,n)$ is related to two-sided vector spaces as follows. Suppose $\phi : K \to M_n(K)$ is a k-central ring homomorphism and K^n is made into a two-sided vector space, K^n_ϕ, via $v \cdot x := v\phi(x)$. Then the K-rational points of the scheme $\mathbb{F}_{\operatorname{im}\phi}(m,n)$ parameterize the two-sided m-dimensional subspaces of K^n_ϕ.

There are other subschemes of $\mathbb{G}(m,n)$ which parameterize two-sided vector spaces as well. In this paper, we study the geometry of $\mathbb{F}_A(m,n)$ and two other subschemes of $\mathbb{G}(m,n)$, $\mathbb{G}_A(m,n)$ and $\mathbb{H}_A(m,n)$, which have the same K-rational points as $\mathbb{F}_A(m,n)$. Our justification for studying $\mathbb{G}_A(m,n)$ is that we are able to give a global description of it as an intersection of $\mathbb{G}(m,n)$ and the projectivization of a vector space (Theorem 3.3). Our justification for studying $\mathbb{H}_A(m,n)$ is that it is a subscheme of $\mathbb{G}_A(m,n)$ which has a smooth, reduced, irreducible open subscheme which covers its K-rational points (Theorem 4.9).

We now describe $\mathbb{G}_A(m,n)$ by its functor of points, $G_A(m,n)$. The R-rational points of this functor are the free rank m summands of R^n, M, which have the property that the image of $\bigwedge^m M$ under the composition

$$\bigwedge^m R^n \xrightarrow{\cong} \bigwedge^m (R \otimes_K K^n) \xrightarrow{\cong} R \otimes_K \bigwedge^m K^n$$

has an R-module generator of the form

$$\sum_i r_i \otimes v_{i1} \wedge \cdots \wedge v_{im}$$

where $\operatorname{Span}_K\{v_{i1}, \ldots, v_{im}\}$ is A-invariant for all i (see Definition 2.3). The motivation behind this definition is that when $A - \{0\} \subset GL_n(K)$ and K^n is homogeneous as a $K \otimes_k A$-module (see Section 2 for a description of the action of $K \otimes_k A$ on K^n), $G_A(m,n)$ solves the same parameterization problem that $F_A(m,n)$ does, in the sense that $G_A(m,n)(K) = F_A(m,n)(K)$. Although it is not clear from the definitions, the functors $G_A(m,n)$ and $F_A(m,n)$ are distinct (Example 6.7).

We prove in Section 3 that $G_A(m,n)$ has a simple global description (Theorem 3.3):

Theorem. *Let*

$$\bigwedge\nolimits^m_A = \operatorname{Span}_K\{v_1 \wedge \cdots \wedge v_m | v_1, \ldots, v_m \text{ is a basis for an } A\text{-invariant subspace of } K^n\}.$$

The functor $G_A(m,n)$ is represented by the pullback of the diagram

$$\begin{array}{ccc} & & \mathbb{P}_K((\bigwedge^m_A)^*) \\ & & \downarrow \\ \mathbb{G}(m,n) & \longrightarrow & \mathbb{P}_K((\bigwedge^m K^n)^*) \end{array}$$

whose horizontal is the canonical embedding, and whose vertical is induced by the inclusion $\bigwedge^m_A \to \bigwedge^m K^n$.

To the authors knowledge, there is no similar description of $\mathbb{F}_A(m,n)$. The functorial description of $\mathbb{G}_A(m,n)$ allows us to describe the tangent space to $\mathbb{G}_A(m,n)$ (Theorem 3.6).

We define $\mathbb{H}_A(m,n)$ to be the pullback of the diagram

$$\begin{array}{ccc} & & \mathbb{G}_A(m,n) \\ & & \downarrow \\ \mathbb{F}_A(m,n) & \longrightarrow & \mathbb{G}(m,n). \end{array}$$

Suppose $S \subset K^n$ is a simple $K \otimes_k A$-module such that $\dim_K S = m$, and K^n is S-homogeneous and semisimple as a $K \otimes_k A$-module. In Section 4, we construct an affine open cover of the K-rational points of $\mathbb{F}_A(m,n)$. Furthermore, when K is infinite and A is commutative, we construct an affine open cover of the K-rational points of $\mathbb{H}_A(m,n)$. As a consequence, we prove the following (Theorem 4.9):

Theorem. *Suppose $K^n \cong S^{\oplus l}$ as $K \otimes_k A$-modules. Then $\mathbb{F}_A(m,n)$ contains an open subscheme which is smooth, reduced, irreducible, of dimension $lm - m$ and has the same K-rational points as $\mathbb{F}_A(m,n)$. Furthermore, if K is infinite and A is commutative, then $\mathbb{H}_A(m,n)$ contains an open subscheme which is smooth, reduced, irreducible, of dimension $lm-m$ and has the same K-rational points as $\mathbb{H}_A(m,n)$.*

Now, let $V = K^n_\phi$ and let W be a two-sided vector space. In Section 6, we use $\mathbb{F}_A(m,n)$, $\mathbb{G}_A(m,n)$, and $\mathbb{H}_A(m,n)$ to construct three parameter spaces of two-sided subspaces of V of rank $[W]$ (see Section 5 for the definition of rank). We denote these parameter spaces by $\mathbb{F}_\phi([W],V)$, $\mathbb{G}_\phi([W],V)$, and $\mathbb{H}_\phi([W],V)$. We then provide examples to show that, although $\mathbb{F}_\phi([W],V)$, $\mathbb{G}_\phi([W],V)$, and $\mathbb{H}_\phi([W],V)$ have the same K-rational points, $\mathbb{F}_\phi([W],V) \neq \mathbb{G}_\phi([W],V)$ and $\mathbb{F}_\phi([W],V) \neq \mathbb{H}_\phi([W],V)$ for certain $[W]$ and V. As a consequence, $\mathbb{F}_A(m,n) \neq \mathbb{G}_A(m,n)$ and $\mathbb{F}_A(m,n) \neq \mathbb{H}_A(m,n)$ for certain A, m, and n.

We then show that, if F is an extension field of K, then every element of $G_\phi([S],V)(F)$ and of $H_\phi([S],V)(F)$ is isomorphic to $F \otimes_K S$ as $F \otimes_k K$-modules (Theorem 6.10).

We conclude by studying the geometry of the parameter spaces $\mathbb{F}_\phi([W],V)$, $\mathbb{G}_\phi([W],V)$, and $\mathbb{H}_\phi([W],V)$ in two cases. In case K/k is finite and Galois, we prove that the three spaces are equal to each other, and equal to the product of Grassmannians (Corollary 6.12). In case K is infinite, $\{S_i\}_{i=1}^r$ consists of nonisomorphic simples with $\dim S_i = m_i$, and V is semisimple with l_i factors of S_i, we prove the following (Corollary 6.15):

Theorem. *$\mathbb{F}_\phi([S_1]+\cdots+[S_r],V)$ and $\mathbb{H}_\phi([S_1]+\cdots+[S_r],V)$ contain smooth, reduced, irreducible open subschemes of dimension $\sum_{i=1}^r l_i m_i - m_i$ which cover their K-rational points.*

Aside from their significance as generalizations of Grassmannians, parameter spaces of two-sided subspaces of V, or Grassmannians of two-sided subspaces of V, are related to classification questions in noncommutative algebraic geometry. The subject of noncommutative algebraic geometry is concerned, among other things, with classifying noncommutative projective surfaces (see [7] for an introduction to this subject). One important class of noncommutative surfaces, the class of noncommutative ruled surfaces, is constructed from noncommutative vector bundles.

Let U and X be schemes, and let X be a U-scheme. By a "U-central noncommutative vector bundle over X", we mean an $\mathcal{O}_U$-central, coherent sheaf $X-X$-bimodule which is locally free on the right and left [8, Definition 2.3, p. 440].

Two-sided vector spaces are related to noncommutative vector bundles as follows. If $\mathcal{E}$ is a noncommutative vector bundle over an integral scheme X, $\mathcal{E}_\eta$ is a two-sided vector space over $k(X)$. In addition, if $U = \operatorname{Spec} k$ and $X = \operatorname{Spec} K$, a U-central noncommutative vector bundle over X is just a two-sided vector space.

Let $\mathcal{E}$ be a (commutative) vector bundle over X. An important problem in algebraic geometry is to parameterize quotients of $\mathcal{E}$ with fixed Hilbert polynomial, and study the resulting parameter space. We are interested in the analogous problem in noncommutative algebraic geometry: to parameterize U-central quotients of a U-central noncommutative vector bundle over X with fixed invariants, and study the resulting parameter space. Thus, the results in this paper address this problem when $U = \operatorname{Spec} k$ and $X = \operatorname{Spec} K$.

Notation and conventions: We let Sets denote the category of sets and $K-\mathsf{alg}$ denote the category of commutative K-algebras. For any scheme Y over $\operatorname{Spec} K$, we let h_Y denote the functor of points of Y, i.e. the functor h_Y from the category $K-\mathsf{alg}$ to the category Sets is the functor $\operatorname{Hom}_{\operatorname{Spec} K}(\operatorname{Spec} -, Y)$. Unless otherwise specified, all unlabeled isomorphisms are assumed to be canonical. Finally, we suppose throughout that $A \subset M_n(K)$ is a k-algebra and R is a commutative K-algebra.

Other notation and conventions will be introduced locally.

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2. Subfunctors of the Grassmannian

Recall that the functor of points of the Grassmannian over $\operatorname{Spec} K$ is the functor $G(m,n) : K-\mathsf{alg} \to \mathsf{Sets}$ defined on R as the set of free rank m summands of R^n, and defined on morphisms as the pullback [2, Exercise VI-18, p. 261].

In this section, we define three subfunctors of $G(m,n)$, $F_A(m,n)$, $G_A(m,n)$, and $H_A(m,n)$. We will see that $F_A(m,n)$ and $H_A(m,n)$ parameterize m-dimensional subspaces of K^n which are invariant under the action of A, and $G_A(m,n)$ does so under suitable hypotheses on A.

Let $m = \sum_{i=1}^n r_i e_i \in R^n$, where e_i is the standard unit vector. We note that the action $r \otimes a \cdot m = \sum_{i=1}^n r r_i e_i a$ makes R^n an $R \otimes_k A$-module. We say that $M \subset R^n$ is *A-invariant* if M is an $R \otimes_k A$-submodule.

Definition 2.1. Suppose m is a nonnegative integer. Let $F_A(m,n)(-) : K-\mathsf{alg} \to \mathsf{Sets}$ denote the assignment defined on the object R as

$$\{M \in G(m,n)(R) | M \text{ is } A\text{-invariant}\}$$

and on morphisms $\delta : R \to T$ as the pullback. That is, $F_A(m,n)(\delta)(M)$ equals the image of the map

$$T \otimes_R M \to T \otimes_R R^n \xrightarrow{\cong} T^n \tag{1}$$

whose left arrow is induced by inclusion $M \subset R^n$.

The proof of the following result is straightforward, so we omit it.

Lemma 2.2. *The assignment $F_A(m,n) : K - \mathsf{alg} \to \mathsf{Sets}$ is a functor.*

We call elements of $F_A(m,n)(R)$ *free rank m A-invariant families over* $\operatorname{Spec} R$, or *free A-invariant families* when m, n and R are understood.

Definition 2.3. Let $M \subset R^n$ be a free rank m summand. We say M is *generated by A-invariants over* $\operatorname{Spec} R$ or is *generated by A-invariants* if R is understood, if $\bigwedge^m M$ maps, under the composition

$$\bigwedge^m M \to \bigwedge^m R^n \xrightarrow{\cong} \bigwedge^m (R \otimes_K K^n) \xrightarrow{\cong} R \otimes_K \bigwedge^m K^n \tag{2}$$

whose left arrow is induced by inclusion, to an R-module with generator of the form

$$\sum_i r_i \otimes v_{i1} \wedge \cdots \wedge v_{im}, \tag{3}$$

where, for all i, $\{v_{i1}, \ldots, v_{im}\}$ is a basis for a rank m A-invariant subspace of K^n.

For a discussion of the motivation behind this definition, see Remark 2.9.

Lemma 2.4. *Let $\delta : R \to T$ be a homomorphism of K-algebras, and let M be a free rank m summand of R^n which is generated by A-invariants over* $\operatorname{Spec} R$. *Then the image of $T \otimes_R M$ under (1) is a free rank m summand of T^n which is generated by A-invariants over* $\operatorname{Spec} T$.

Proof. Suppose M has basis $w_1, \ldots, w_m \in R^n$, and $w_1 \wedge \cdots \wedge w_m$ maps to

$$\sum_i r_i \otimes v_{i1} \wedge \cdots \wedge v_{im}$$

under (2). Then $1 \otimes w_1, \ldots, 1 \otimes w_m \in T \otimes_R M$ are generators of $T \otimes_R M$, and, if we let $\overline{w_1}, \ldots, \overline{w_m}$ denote the images of $1 \otimes w_1, \ldots, 1 \otimes w_m$ under (1), then $\overline{w_1}, \ldots, \overline{w_m}$ generates the image of $T \otimes_R M$ under (1). We claim $\overline{w_1} \wedge \cdots \wedge \overline{w_m} \in \bigwedge^m T^n$ maps to $\sum_i \delta(r_i) \otimes v_{i1} \wedge \cdots \wedge v_{im}$ under the composition

$$\bigwedge^m T^n \xrightarrow{\cong} \bigwedge^m (T \otimes_K K^n) \xrightarrow{\cong} T \otimes_K \bigwedge^m K^n. \tag{4}$$

To prove the claim, we first note that a straightforward computation implies that

$$\begin{tikzcd}
\bigwedge^m (T \otimes_R R^n) \arrow[r, "\cong"] & \bigwedge^m T^n \arrow[r, "\cong"] & \bigwedge^m (T \otimes_K K^n) \arrow[r, "\cong"] & T \otimes_K \bigwedge^m K^n \\
T \otimes \bigwedge^m R^n \arrow[u, "\cong"] \arrow[r, "\cong"] & T \otimes_R \bigwedge^m (R \otimes_R K^n) \arrow[r, "\cong"] & T \otimes_R (R \otimes_K \bigwedge^m K^n) \arrow[r, "\cong"] & T \otimes_K \bigwedge^m K^n \arrow[u, "="']
\end{tikzcd} \tag{5}$$

commutes (recall our convention about unlabeled isomorphisms). Furthermore, the image of $1 \otimes w_1 \wedge \cdots \wedge w_m \in T \otimes_R \bigwedge^m R^n$ under the right-hand route of (5) equals $\sum_i \delta(r_i) \otimes v_{i1} \wedge \cdots \wedge v_{im}$. Therefore, the image of $1 \otimes w_1 \wedge \cdots \wedge w_m \in T \otimes_R \bigwedge^m R^n$ under the left-hand route of (5) equals $\sum_i \delta(r_i) \otimes v_{i1} \wedge \cdots \wedge v_{im}$. Finally, the image of $1 \otimes w_1 \wedge \cdots \wedge w_m \in T \otimes_R \bigwedge^m R^n$ under the first two maps of the left-hand route of (5) equals $\overline{w_1} \wedge \cdots \wedge \overline{w_m} \in \bigwedge^m T^n$. The claim, and hence the lemma, follows from the fact that the composition of the third and fourth arrows of the left-hand route of (5) is the composition (4). $\square$

Definition 2.5. Let $G_A(m,n)(-) : K-\mathsf{alg} \to \mathsf{Sets}$ denote the assignment defined on the object R as

$$\{M \in G(m,n)(R) | M \text{ is generated by } A\text{-invariants}\}$$

and on morphisms as the pullback.

The next result follows immediately from Lemma 2.4.

Lemma 2.6. *$G_A(m,n) : K-\mathsf{alg} \to \mathsf{Sets}$ is a functor.*

We call elements of $G_A(m,n)(R)$ *free rank m families generated by A-invariants over* $\operatorname{Spec} R$, or *free families generated by A-invariants* when m, n and R are understood.

Remark 2.7. It follows immediately from Definition 2.3 that

$$F_A(m,n)(K) \subset G_A(m,n)(K).$$

We now find sufficient conditions under which $F_A(m,n)(K) = G_A(m,n)(K)$.

Lemma 2.8. *Let M be a free rank m family generated by A-invariants over* $\operatorname{Spec} R$. *If $M_{\mathfrak{m}}$ is A-invariant for every maximal ideal $\mathfrak{m}$ of R, then M is A-invariant. If R is a field, $A-\{0\} \subset GL_n(K)$, and K^n is homogeneous as a $K \otimes_k A$-module, then M is A-invariant.*

Proof. Suppose $\mathfrak{m}$ is a maximal ideal of R, a is an element of A, and $N = Ma + M$. The diagram

$$\begin{array}{ccccc} R_{\mathfrak{m}} \otimes_R M & \to & R_{\mathfrak{m}} \otimes_R R^n & \xrightarrow{\cong} & R^n_{\mathfrak{m}} \\ \downarrow & & \downarrow = & & \\ R_{\mathfrak{m}} \otimes_R N & \to & R_{\mathfrak{m}} \otimes_R R^n & \xrightarrow{\cong} & R^n_{\mathfrak{m}} \end{array}$$

whose left horizontals and left vertical are induced by inclusion, commutes, and the left horizontals are injective since localization is exact. By Lemma 2.4, the image, $\overline{M}$, of the top horizontal composition is generated by A-invariants. Thus, by hypothesis, $\overline{M}$ is A-invariant. Hence, the left vertical is surjective, and so the map

$$M_{\mathfrak{m}} \to N_{\mathfrak{m}}$$

induced by inclusion is an epimorphism. It follows from [1, Corollary 2.9, p. 68] that $M = N$, and hence that M is A-invariant.

Next, suppose R is a field, $A-\{0\} \subset GL_n(K)$, K^n is homogeneous as a $K \otimes_k A$-module, and M has basis $w_1, \ldots, w_m \in R^n$. If M were not A-invariant, then there would exist an $1 \leq i \leq m$ and an $a \in A$ such that $w_i a$ is not an element of M. Thus, since a is invertible, $w_1 a \wedge \cdots \wedge w_m a$ would not be proportional to $w_1 \wedge \cdots \wedge w_m$.

On the other hand, since K^n is homogeneous, the determinant of the matrix corresponding to $a \in A$ acting on an m-dimensional, A-invariant, subspace V is independent of V. Thus, since M is generated by A-invariants, $w_1 a \wedge \cdots \wedge w_m a = c w_1 \wedge \cdots \wedge w_m$ for some nonzero $c \in K$. We conclude that M is A-invariant. □

Remark 2.9. As a consequence of Lemma 2.8, if $A - \{0\} \subset GL_n(K)$, and K^n is homogeneous as a $K \otimes_k A$-module,

$$G_A(m,n)(K) = F_A(m,n)(K).$$

Thus, these two functors parameterize the same object. On the other hand, we will see in Theorem 3.3 that the scheme representing $G_A(m,n)$ has a simple global description as a pullback of $\mathbb{G}(m,n)$ and the projectivization of a vector space. These two facts provide motivation for Definition 2.3.

Definition 2.10. Let $H_A(m,n)(-) : K-\mathsf{alg} \to \mathsf{Sets}$ denote the fibered product of functors $F_A(m,n) \times_{G(m,n)} G_A(m,n)$ induced by inclusion of $F_A(m,n)$ and $G_A(m,n)$ in $G(m,n)$ [2, Definition VI-4, p. 254].

We call elements of $H_A(m,n)(R)$ *free rank m A-invariant families generated by A-invariants over* $\operatorname{Spec} R$, or *free A-invariant families generated by A-invariants* when m, n, and R are understood.

Remark 2.11. It follows from Remark 2.7 that $H_A(m,n)(K) = F_A(m,n)(K)$.

3. Representability of $G_A(m,n)$ and $H_A(m,n)$

It was proven in [3] that $F_A(m,n)$ is representable by a subscheme of the Grassmannian $\mathbb{G}(m,n)$. The main result of this section is that $G_A(m,n)$ is representable by the intersection of $\mathbb{G}(m,n)$ and the projectivization of a vector space. It will follow easily that $H_A(m,n)$ is representable as well. We conclude the section by computing the tangent space to $G_A(m,n)$.

Let $\mathbb{P}(-)$ denote the projectivization functor. That is, if M is a K-module, we let $\mathbb{P}(M)$ denote the scheme whose R-rational points equal equivalence classes of epimorphisms $\tau : R \otimes_K M \to L$, where L is an invertible R-module, such that $\tau_1 : R \otimes_K M \to L_1$ is equivalent to $\tau_2 : R \otimes_K M \to L_2$ iff there exists an isomorphism $\psi : L_1 \to L_2$ such that $\tau_2 = \psi\tau_1$.

Before proving that $G_A(m,n)$ is representable, we recall two preliminary facts.

Lemma 3.1. *Let U be a subspace of $\bigwedge^m K^n$. There is a natural isomorphism*

$$R \otimes_K (U)^* \xrightarrow{\cong} (R \otimes_K U)^*,$$

and the canonical isomorphism $\bigwedge^m R^n \longrightarrow R \otimes_K \bigwedge^m K^n$ induces an isomorphism

$$(R \otimes_K \textstyle\bigwedge^m K^n)^* \xrightarrow{\cong} (\bigwedge^m R^n)^*.$$

We omit the straightforward proof of the next result.

Lemma 3.2. *Let F denote the full subcategory of the category of R-modules consisting of finitely generated free R-modules. Then the functor*

$$\operatorname{Hom}_R(-,R) : \mathsf{F} \to \mathsf{F}$$

is full and faithful.

We let

$$\textstyle\bigwedge_A^m = \operatorname{Span}_K\{v_1 \wedge \cdots \wedge v_m | v_1, \ldots, v_m \text{ is a basis for an } A\text{-invariant subspace of } V\}.$$

Theorem 3.3. *The functor $G_A(m,n)$ is represented by the pullback of the diagram*

$$\begin{array}{ccc} & & \mathbb{P}((\bigwedge_A^m)^*) \\ & & \downarrow \\ \mathbb{G}(m,n) & \longrightarrow & \mathbb{P}((\bigwedge^m K^n)^*) \end{array} \qquad (6)$$

whose horizontal is the canonical embedding, and whose vertical is induced by the inclusion $\bigwedge_A^m \to \bigwedge^m K^n$.

Proof. By [2, p. 260], it suffices to prove that the functor $G_A(m,n)$ is the pullback of functors

$$h_{\mathbb{G}(m,n)} \times_{h_{\mathbb{P}((\bigwedge^m K^n)^*)}} h_{\mathbb{P}((\bigwedge_A^m)^*)} \tag{7}$$

induced by (6).

Let $M \subset R^n$ be a free rank m summand. We recall a preliminary fact. The map $\bigwedge^m M \to \bigwedge^m R^n$ induced by the inclusion of M in R^n identifies $\bigwedge^m M$ with a summand of $\bigwedge^m R^n$. Thus, the induced map

$$\psi : (\bigwedge^m R^n)^* \longrightarrow (\bigwedge^m M)^*$$

is an epimorphism.

We now prove that $G_A(m,n)$ equals (7). We note that a free rank m summand $M \subset R^n$ is an R-rational point of (7) iff the epimorphism

$$R \otimes_K (\bigwedge^m K^n)^* \xrightarrow{\cong} (\bigwedge^m R^n)^* \xrightarrow{\psi} (\bigwedge^m M)^* \tag{8}$$

whose left arrow is the map from Lemma 3.1, factors through the map

$$R \otimes_K (\bigwedge^m K^n)^* \longrightarrow R \otimes_K (\bigwedge_A^m)^* \tag{9}$$

induced by the inclusion $\bigwedge_A^m \subset \bigwedge^m K^n$. Thus, to prove the result, it suffices to show that (8) factors through (9) iff M is generated by A-invariants. Now, (8) factors through (9) iff there exists a map $\gamma^* : (R \otimes_K \bigwedge_A^m)^* \longrightarrow (\bigwedge^m M)^*$ making the diagram

$$\begin{tikzcd}
R \otimes_K (\bigwedge^m K^n)^* \arrow[r, "\cong"] \arrow[d] & (R \otimes_K \bigwedge^m K^n)^* \arrow[r, "\cong"] \arrow[d] & (\bigwedge^m R^n)^* \arrow[d, "\psi"] \\
R \otimes_K (\bigwedge_A^m)^* \arrow[r, "\cong"] & (R \otimes_K \bigwedge_A^m)^* \arrow[r, "\gamma^*"] & (\bigwedge^m M)^*
\end{tikzcd} \tag{10}$$

whose left and middle vertical are induced by inclusion, and whose top horizontals and bottom left horizontal are from Lemma 3.1, commute. By Lemma 3.1, the left square of (10) commutes. Thus, by Lemma 3.2, there exists a map γ^* making (10) commute iff there exists a map $\gamma : \bigwedge^m M \to R \otimes_K \bigwedge_A^m$ making the diagram

$$\begin{tikzcd}
R \otimes_K \bigwedge^m K^n & \bigwedge^m R^n \arrow[l, "\cong"'] \\
R \otimes_K \bigwedge_A^m \arrow[u] & \bigwedge^m M \arrow[l, "\gamma"'] \arrow[u]
\end{tikzcd} \tag{11}$$

whose verticals are inclusions, commute. This occurs iff M is generated by A-invariants, i.e. iff $M \in G_A(m,n)(R)$. $\square$

We denote the pullback of (6) by $\mathbb{G}_A(m,n)$. The following result is now immediate:

Corollary 3.4. *$\mathbb{G}_A(m,n)$ is a projective subscheme of $\mathbb{G}(m,n)$.*

We also note that $H_A(m,n)$ is representable:

Corollary 3.5. *$H_A(m,n)$ is represented by the pullback of the diagram*

$$\begin{tikzcd}
 & \mathbb{G}_A(m,n) \arrow[d] \\
\mathbb{F}_A(m,n) \arrow[r] & \mathbb{G}(m,n).
\end{tikzcd}$$

whose arrows are induced by the inclusion of functors $G_A(m,n) \subset G(m,n)$ *and* $F_A(m,n) \subset G(m,n)$.

Proof. Since $H_A(m,n)$ is defined as the fibered product $F_A(m,n) \times_{G(m,n)} G_A(m,n)$ induced by the inclusion of functors $G_A(m,n) \subset G(m,n)$ and $F_A(m,n) \subset G(m,n)$, the result follows immediately from [2, p. 260]. $\square$

We end the section by computing the Zariski tangent space to $\mathbb{G}_A(m,n)$ at the K-rational point $E = \operatorname{Span}_K\{e_1, \dots, e_m\} \in G_A(m,n)(K)$. Recall that if

$$\Psi : G_A(m,n)(K[\epsilon]/(\epsilon^2)) \to G_A(m,n)(K)$$

is the map induced by the quotient $K[\epsilon]/(\epsilon^2) \to K$ sending ϵ to 0, the Zariski tangent space to $\mathbb{G}_A(m,n)$ at the K-rational point $E \in G_A(m,n)(K)$ is the set

$$T_E = \{M \in G_A(m,n)(K[\epsilon]/(\epsilon^2)) | \Psi(M) = E\}$$

with vector space structure defined as follows: Suppose $\{f_i\}_{i=1}^m$ and $\{g_i\}_{i=1}^m$ are subsets of K^n. If $M \in T_E$ has basis $\{e_i + \epsilon f_i\}_{i=1}^m$, $M' \in T_E$ has basis $\{e_i + \epsilon g_i\}_{i=1}^m$, and $a, b \in K$, we let $aM + bM' \in T_E$ denote the family with basis $\{e_i + \epsilon(af_i + bg_i)\}$. It is straightforward to check that the vector space structure is independent of choices made.

We define a map

$$d : \operatorname{Hom}_K(E, K^n) \to \operatorname{Hom}_K(\textstyle\bigwedge^m E, \bigwedge^m K^n)$$

as follows. For $\psi \in \operatorname{Hom}_K(E, K^n)$, we define $d(\psi)$ on totally decomposable wedges as

$$d(\psi)(e_1 \wedge \cdots \wedge e_m) = \sum_{i=1}^m e_1 \wedge \cdots \wedge e_{i-1} \wedge \psi(e_i) \wedge e_{i+1} \wedge \cdots \wedge e_m$$

and extend linearly. It is straightforward to check that d is K-linear.

Theorem 3.6. *Suppose* $V = E \oplus L$ *as a* K*-module for some* K*-submodule* L *of* K^n. *The tangent space to* $\mathbb{G}_A(m,n)$ *at* $E \in \mathbb{G}_A(m,n)(K)$ *is isomorphic to*

$$S_E = \{\psi \in \operatorname{Hom}_K(E, L) | \operatorname{im} d(\psi) \subset \textstyle\bigwedge_A^m\}.$$

Proof. We define a map

$$\Phi : T_E \to S_E$$

as follows: let $M \in T_E$ have basis $\{e_i + \epsilon(s_i + t_i)\}_{i=1}^m$ where $\{e_1, \dots, e_m\}$ is a basis for E, $s_i \in E$, and $t_i \in L$. Define $\psi \in \operatorname{Hom}_K(E, K^n)$ by $\psi(e_i) = t_i$. We let $\Phi(M) = \psi$, and we omit the straightforward proof of the fact that the definition of Φ is independent of choices made.

Step 1: We prove Φ *is a well defined map of vector spaces.* We omit the straightforward proof of the fact that as a map to $\operatorname{Hom}_K(E, K^n)$, Φ is K-linear. It remains to show that $\Phi(M) \in S_E$, i.e. that $\operatorname{im} d(\psi) \subset \bigwedge_A^m$. Since M is generated by A-invariants, $(e_1 + \epsilon(s_1 + t_1)) \wedge \cdots \wedge (e_m + \epsilon(s_m + t_m)) \in \bigwedge^m M$ maps, under (2) to

$$\sum_i r_i \otimes v_{i_1} \wedge \cdots \wedge v_{i_m}, \tag{12}$$

where $r_i \in K[\epsilon]/(\epsilon)^2$ and $v_{i_1} \wedge \cdots \wedge v_{i_m} \in \bigwedge_A^m$. Thus,

$$(s_1 + t_1) \wedge e_2 \wedge \cdots \wedge e_m + \cdots + e_1 \wedge \cdots \wedge e_{m-1} \wedge (s_m + t_m) = \sum_j a_j w_{j_1} \wedge \cdots \wedge w_{j_m},$$

where $a_j \in K$ and $w_{j_1} \wedge \cdots \wedge w_{j_m} \in \bigwedge_A^m$. This implies
(13)
$$s_1 \wedge e_2 \wedge \cdots \wedge e_m + \cdots + e_1 \wedge \cdots \wedge e_{m-1} \wedge s_m + t_1 \wedge e_2 \wedge \cdots \wedge e_m + \cdots + e_1 \wedge \cdots \wedge e_{m-1} \wedge t_m$$
is in $\bigwedge_A^m$. Since $s_i \in E$, each of the first m terms of (13) is either 0 or a multiple of $e_1 \wedge \cdots \wedge e_m$, which is in $\bigwedge_A^m$. Thus,
$$t_1 \wedge e_2 \wedge \cdots \wedge e_m + \cdots + e_1 \wedge \cdots \wedge e_{m-1} \wedge t_m \in \bigwedge_A^m,$$
i.e. $d(\psi)(e_1 \wedge \cdots \wedge e_m) \in \bigwedge_A^m$ as desired.

Step 2: We prove Φ is one-to-one and onto. If $\Phi(M) = 0$ then M has a basis $\{e_i\}_{i=1}^m$, and thus M is the identity element of T_E. This establishes the fact that Φ is one-to-one.

Let $\psi \in S_E$, and let $M \in T_E$ have basis $\{e_i + \epsilon\psi(e_i)\}_{i=1}^m$. To prove Φ is onto, we must prove that M is generated by A-invariants. By hypothesis,
$$\psi(e_1) \wedge e_2 \wedge \cdots \wedge e_m + \cdots + e_1 \wedge \cdots \wedge e_{m-1} \wedge \psi(e_m) = \sum_j b_j u_{j_1} \wedge \cdots \wedge u_{j_m},$$
where $b_j \in K$ and $u_{j_1} \wedge \cdots \wedge u_{j_m} \in \bigwedge_A^m$, and thus the image of $(e_1 + \epsilon\psi(e_1)) \wedge \cdots \wedge (e_m + \epsilon\psi(e_m))$ maps, under (2), to
$$\sum_i r_i v_{i_1} \wedge \cdots \wedge v_{i_m},$$
where $r_i \in K[\epsilon]/(\epsilon)^2$ and $v_{i_1} \wedge \cdots \wedge v_{i_m} \in \bigwedge_A^m$. Hence, $M \in T_E$, as desired. $\square$

4. Affine open subschemes of $\mathbb{F}_A(m,n)$ and $\mathbb{H}_A(m,n)$

Suppose $S \subset K^n$ is a simple $K \otimes_k A$-module such that $\dim_K S = m$. In this section, we assume K^n is S-homogeneous and semisimple as a $K \otimes_k A$-module. We study the subspace $\bigwedge_A^m \subset \bigwedge^m K^n$ in order to find conditions under which a free rank m A-invariant family is generated by A-invariants. We use our results in order to construct affine open subschemes of $\mathbb{F}_A(m,n)$ and $\mathbb{H}_A(m,n)$ which cover their K-rational points.

We suppose $K^n = S^{\oplus l}$, and we let $\pi_i : R^{lm} \to R^m$ denote projection onto the $(i-1)m+1$ through the imth coordinates.

Lemma 4.1. *Suppose $M \subset R^{lm}$ is A-invariant. If M is principally generated as an $R \otimes_k A$-module by f, and if $\pi_i(f) \in K^m$ for some $1 \leq i \leq l$, then M is a free rank m summand of R^{lm}, and M is generated, as an R-module, by $fa_1, \ldots, fa_m$ for some $a_1, \ldots, a_m \in A$.*

Proof. Suppose $\pi_i(f) = v \in K^m$. Since K^n is S-homogeneous, S is simple, and $\dim_K S = m$, there exist $a_1, \ldots, a_m \in A$ such that
$$\{va_1, \ldots, va_m\}$$
is independent over K. Thus, the R-module generated by $fa_1, \ldots, fa_m$, which we denote by $\langle fa_1, \ldots, fa_m\rangle$, is a free rank m summand of R^{lm}. To complete the proof of the lemma, it suffices to prove that $\langle fa_1, \ldots, fa_m\rangle$ is A-invariant. To this end, we first prove $\pi_i|_M$ is injective. Suppose $\pi_i(xf) = 0$ for $x \in R \otimes_k A$. Since $K^n = S^{\oplus l}$, $\pi_i(xf) = x\pi_i(f)$. Thus, $xv = 0$ in $R \otimes_K S$, so that $x \in \operatorname{ann} R \otimes_K S$. Thus $x \in \operatorname{ann} R \otimes_K V$, again since $K^n = S^{\oplus l}$, so that $xf = 0$.

Now, suppose $a \in A$. We prove $fa \in \langle fa_1, \ldots, fa_m \rangle$. For,

$$\begin{aligned} \pi_i(fa) &= \pi_i(f)a \\ &= va \\ &= b_1va_1 + \cdots + b_mva_m \\ &= \pi(b_1fa_1 + \cdots b_mfa_m), \end{aligned}$$

where $b_1, \ldots, b_m \in K$. Since $\pi_i|_M$ is injective, we must have $fa = b_1fa_1 + \cdots + b_mfa_m$, and the assertion follows. □

For the remainder of Section 4, we suppose $w_i \in S^{\oplus l}$ has one nonzero projection, $v_i \in S$, and $a_1, \ldots, a_m \in A$ are such that

$$\{v_1a_1, \ldots, v_1a_m\}$$

is independent (such $a_1, \ldots, a_m$ exist since S is simple).

Lemma 4.2. *Suppose A is commutative, $b_1, \ldots, b_r \in K$, and consider the set*

$$\{b_1w_1a_1 + \cdots + b_rw_ra_1, \ldots, b_1w_1a_m + \cdots + b_rw_ra_m\}. \tag{14}$$

If (14) is nonzero, then (14) is a basis for an A-invariant subspace of K^n of rank m.

Proof. We first note that, if $u \in S$ and $u \neq 0$, then $\{ua_1, \ldots, ua_m\}$ is independent iff $\{v_1a_1, \ldots, v_1a_m\}$ is independent. For, since S is a simple $K \otimes_k A$-module, there exists an $r \in K \otimes_k A$ such that $ru = v_1$. Thus, since $K \otimes_k A$ is commutative, any dependence relation among $ua_1, \ldots, ua_m$ is a dependence relation among $v_1a_1, \ldots, v_1a_m$ and conversely. Since we have assumed above that $\{v_1a_1, \ldots, v_1a_m\}$ is independent, we may conclude that if $u \in S$ is nonzero then $\{ua_1, \ldots, ua_m\}$ is independent.

Suppose (14) is nonzero. The fact that the set (14) is independent follows from the fact that some projection of (14) to a summand S of $K^n = S^{\oplus l}$ is independent by the argument in the first paragraph. To prove that the K-module generated by (14) is A-invariant, we note that the K-module generated by (14) is contained in the $K \otimes_k A$-module, M, generated by $b_1w_1 + \cdots + b_rw_r$. On the other hand, by Lemma 4.1, M is a free rank m summand of K^{ml}. Since the K-module generated by (14) is a free rank m summand of K^n, it must equal M. □

Proposition 4.3. *Let*

$$I = \{\mathbf{n} = (n_1, \cdots, n_r) \in \mathbb{Z}^r_{\geq 0} | n_1 + \cdots + n_r = m\},$$

let $\{n_1 \cdot 1, \ldots, n_r \cdot r\}$ denote the multiset with n_i copies of i, and let

$$w_{\mathbf{n}} = \sum_{\{(s_1, \ldots, s_m) | \{s_i\}_{i=1}^m = \{n_1 \cdot 1, \ldots, n_r \cdot r\}\}} w_{s_1}a_1 \wedge \cdots \wedge w_{s_m}a_m.$$

If K is infinite, then $w_{\mathbf{n}}$ is an element of $\bigwedge_A^m$ for all $\mathbf{n} \in I$.

Proof. If $r = 1$, then $w_{\mathbf{n}} = w_1a_1 \wedge \cdots \wedge w_1a_m$, and the result follows from Lemma 4.2.

Now suppose $r \geq 2$, so that $|I| = D = \binom{m+r-1}{m} \geq 2$. We begin the proof of the case $r \geq 2$ with two preliminary observations. First, each choice of $[(c_1, \ldots, c_r)] \in \mathbb{P}_K^{r-1}$ corresponds, via the mth Veronese map ν_m, to a point in $\mathbb{P}_K^{D-1}$. Since K is infinite, no $D-2$ plane of $\mathbb{P}_K^{D-1}$ contains the image of ν_m.

If $\mathbf{b} \in K^r$ with $\mathbf{b} = (b_1, \ldots, b_r)$, we let $b^{\mathbf{n}} = b_1^{n_1} \cdots b_r^{n_r}$. Our second preliminary observation is that, for each $i_0 \leq D$, there exist $\mathbf{b}_1, \ldots, \mathbf{b}_{i_0} \in K^r$ with $\mathbf{b}_i = (b_{i1}, \ldots, b_{ir})$, such that $\{(b_i^{\mathbf{n}})_{\mathbf{n} \in I}\}_{i=1}^{i_0} \subset K^D$ is independent. We prove this by induction on $i_0 \geq 1$. The case $i_0 = 1$ is trivial. Now suppose the result holds for $1 \leq i_0$, where $i_0 < D$. Then there exist $\mathbf{b}_1, \ldots, \mathbf{b}_{i_0} \subset K^r$ such that $\{(b_i^{\mathbf{n}})_{\mathbf{n} \in I}\}_{i=1}^{i_0} \subset K^D$ is independent. Thus, since $\nu_m([\mathbf{b}_i]) = [(b_i^{\mathbf{n}})_{\mathbf{n} \in I}]$, the subspace of $\mathbb{P}_K^{D-1}$ spanned by $\nu_m([\mathbf{b}_1]), \ldots, \nu_m([\mathbf{b}_{i_0}])$ is an $i_0 - 1$-plane. Since $i_0 < D$, the argument of the first paragraph implies there exists a $\mathbf{b}_{i_0+1} \subset K^r$ such that $\nu_m([\mathbf{b}_{i_0+1}])$ is not contained in the $i_0 - 1$-plane spanned by $\nu_m([\mathbf{b}_1]), \ldots, \nu_m([\mathbf{b}_{i_0}])$. Thus, $\{(b_i^{\mathbf{n}})_{\mathbf{n} \in I}\}_{i=1}^{i_0+1} \subset K^D$ is independent, as desired. We conclude that there exist $\mathbf{b}_1, \ldots, \mathbf{b}_D \in K^r$, such that $\{(b_i^{\mathbf{n}})_{\mathbf{n} \in I}\}_{i=1}^{D} \subset K^D$ is independent.

We now prove the proposition. For all $1 \leq i \leq D$, the vector

$$(b_{i1} w_1 a_1 + \cdots + b_{ir} w_r a_1) \wedge \cdots \wedge (b_{i1} w_1 a_m + \cdots + b_{ir} w_r a_m) \tag{15}$$

is A-invariant by Lemma 4.2. Thus, by Remark 2.7, (15) is an element of $\bigwedge_A^m$. In addition, (15) equals $\sum_{\mathbf{n} \in I} b_i^{\mathbf{n}} w_{\mathbf{n}}$. Thus, it suffices to prove that $w_{\mathbf{n}} \in \operatorname{Span}\{\sum_{\mathbf{n} \in I} b_i^{\mathbf{n}} w_{\mathbf{n}}\}_{i=1}^D$ for all $\mathbf{n} \in I$. To prove this, we note that since $\{(b_i^{\mathbf{n}})_{\mathbf{n} \in I}\}_{i=1}^{D} \subset K^D$ is independent, $\{\sum_{\mathbf{n} \in I} b_i^{\mathbf{n}} w_{\mathbf{n}}\}_{i=1}^D$ is a set of D independent vectors in $\operatorname{Span}\{w_{\mathbf{n}} | \mathbf{n} \in I\}$. Since $|I| = D$, $\{\sum_{\mathbf{n} \in I} b_i^{\mathbf{n}} w_{\mathbf{n}}\}_{i=1}^D$ forms a basis for $\operatorname{Span}\{w_{\mathbf{n}} | \mathbf{n} \in I\}$. Thus, $w_{\mathbf{n}} \in \operatorname{Span}\{\sum_{\mathbf{n} \in I} b_i^{\mathbf{n}} w_{\mathbf{n}}\}_{i=1}^D$ for all $\mathbf{n} \in I$, and the proof of the proposition follows. □

Corollary 4.4. *Suppose K is infinite, A is commutative, and $M \subset R^{lm}$ is A-invariant. If M is principally generated as an $R \otimes_k A$-module by f, and if $\pi_i(f) \in K^m$ for some $1 \leq i \leq l$, then M is a free rank m family generated by A-invariants over* $\operatorname{Spec} R$.

Proof. Throughout this proof, we let $[p]$ denote the set $\{1, \ldots, p\}$. Let $\pi_i(f) = v \in K^m$. By Lemma 4.1, there exist $a_1, \ldots, a_m \in A$ such that M is a free rank m summand of R^{lm} and M is generated as an R-module by $fa_1, \ldots, fa_m$. Thus, $\bigwedge^m M$ is generated by $fa_1 \wedge \cdots \wedge fa_m$ as an R-module. On the other hand, $f = x_1 u_1 + \cdots + x_l u_l$ where $u_i = (0, \cdots, 0, v_i, 0, \cdots, 0) \in S^{\oplus l}$ has ith nonzero projection to S, and

$$x_i = \sum_{j=1}^n r_{ij} \otimes b_{ij} \in R \otimes_k A.$$

Thus, $\bigwedge^m M$ is generated by $(\sum_{i=1}^l x_i u_i) a_1 \wedge \cdots \wedge (\sum_{i=1}^l x_i u_i) a_m$ which equals

$$\sum_{(i_1, \ldots, i_m) \in [l]^m} x_{i_1} u_{i_1} a_1 \wedge \cdots \wedge x_{i_m} u_{i_m} a_m.$$

Expanding further, we find the expression above equals

$$\sum_{(i_1, \ldots, i_m) \in [l]^m} \left(\sum_{(j_1, \ldots, j_m) \in [n]^m} (r_{i_1 j_1} \otimes b_{i_1 j_1}) \cdot u_{i_1} a_1 \wedge \cdots \wedge (r_{i_m j_m} \otimes b_{i_m j_m}) \cdot u_{i_m} a_m \right),$$

which equals

$$\sum_J r_{i_1 j_1} \cdots r_{i_m j_m} u_{i_1} b_{i_1 j_1} a_1 \wedge \cdots \wedge u_{i_m} b_{i_m j_m} a_m. \tag{16}$$

where $J = ([l] \times [n])^m$. In order to prove M is generated by A-invariants, it suffices to prove (16) is an element in the image of the composition

$$R \otimes_K \bigwedge\nolimits_A^m \to R \otimes_K \bigwedge\nolimits^m K^{lm} \xrightarrow{\cong} \bigwedge\nolimits^m R^{lm}, \tag{17}$$

whose left arrow is induced by inclusion.

Let S_m denote the mth symmetric group. We note that S_m acts on J via $\sigma \cdot ((i_1, j_1), \ldots, (i_m, j_m)) = ((i_{\sigma(1)}, j_{\sigma(1)}), \ldots, (i_{\sigma(m)}, j_{\sigma(m)}))$, and so J is partitioned into the orbits of this action. Thus, in order to prove (16) is an element in the image of (17), it suffices to show

$$w = \sum_{\sigma \in \mathrm{S}_m} u_{i_{\sigma(1)}} b_{i_{\sigma(1)} j_{\sigma(1)}} a_1 \wedge \cdots \wedge u_{i_{\sigma(m)}} b_{i_{\sigma(m)} j_{\sigma(m)}} a_m$$

is an element of $\bigwedge_A^m$, since (16) is an R-linear combination of images of terms of the form $1 \otimes_K w$ under (17). If we let $w_q = u_{i_q} b_{i_q j_q}$ for $1 \leq q \leq m$, and we let $\mathbf{n} = (1, \ldots, 1) \in \mathbb{Z}_{\geq 0}^m$, w is of the form $w_{\mathbf{n}}$ (defined in Proposition 4.3). Since K is infinite, the corollary follows from Proposition 4.3. $\square$

We end this section by constructing collections of affine open subfunctors of $F_A(m, n)$ and $H_A(m, n)$ which cover their K-rational points.

For the remainder of this section, B will denote the K-algebra $K[x_1, \ldots, x_{lm-m}]$, and

$$\langle (r_1, \ldots, r_{lm}) \rangle \subset R^{lm}$$

will denote the $R \otimes_k A$-submodule of R^{lm} generated by $(r_1, \ldots, r_{lm})$. We will abuse notation as follows: if C and D are K-algebras and $\psi \in h_{\operatorname{Spec} C}(D)$, we let $\psi : C \to D$ denote the induced map of rings.

For each $1 \leq i \leq l$, and each R, we define a map

$$\Phi_{iR} : h_{\operatorname{Spec} B}(R) \to G(m, n)(R)$$

as follows: if $\psi \in h_{\operatorname{Spec} B}(R)$, let

$$\Phi_{iR}(\psi) = \langle (\psi(x_1), \ldots, \psi(x_{(i-1)m}), 1, 0, \ldots, 0, \psi(x_{(i-1)m+1}), \ldots, \psi(x_{lm-m})) \rangle. \tag{18}$$

Lemma 4.5. *Φ_{iR} is a well defined map of sets, and induces a natural transformation $\Phi_i : h_{\operatorname{Spec} B} \to G(m, n)$ which factors through the inclusion $F_A(m, n) \to G(m, n)$. Furthermore, if K is infinite and A is commutative, then Φ_i factors through the inclusion $H_A(m, n) \to G(m, n)$.*

Proof. Suppose $\psi \in h_{\operatorname{Spec} B}(R)$. By Lemma 4.1, $\Phi_{iR}(\psi)$ is a free rank m A-invariant submodule of R^{lm}, whence the first assertion. The proof that Φ_{iR} induces a natural transformation $\Phi_i : h_{\operatorname{Spec} B} \to G(m, n)$ follows from a routine computation, which we omit. Since $\Phi_{iR}(\psi)$ is A-invariant, Φ_i factors through the inclusion $F_A(m, n) \to G(m, n)$. If K is infinite and A is commutative, Corollary 4.4 implies that $\Phi_{iR}(\psi)$ is generated by A-invariants. Thus, Φ_i factors through the inclusion $H_A(m, n) \to G(m, n)$. $\square$

We abuse notation by denoting both factors in the above lemma by Φ_i.

Lemma 4.6. *$\Phi_i : h_{\operatorname{Spec} B} \to F_A(m, n)$ is an open subfunctor. Furthermore, if K is infinite and A is commutative, then $\Phi_i : h_{\operatorname{Spec} B} \to H_A(m, n)$ is an open subfunctor.*

Proof. Suppose $\Psi : h_{\operatorname{Spec} R} \to F_A(m,n)$ is a natural transformation. By Lemma 4.5, we must prove that, if $h_{\operatorname{Spec} B,\Psi}$ is the pullback in the diagram

$$\begin{tikzcd} h_{\operatorname{Spec} B,\Psi} \arrow[r, "\Gamma"] \arrow[d] & h_{\operatorname{Spec} R} \arrow[d, "\Psi"] \\ h_{\operatorname{Spec} B} \arrow[r, "\Phi_i"'] & F_A(m,n) \end{tikzcd}$$

then the induced map $\Gamma : h_{\operatorname{Spec} B,\Psi} \to h_{\operatorname{Spec} R}$ corresponds to the inclusion of an affine open subscheme of $\operatorname{Spec} R$.

By [2, Exercise VI-6, p. 254], this is equivalent to showing that there exists some ideal I of R such that, for any K-algebra T,

$$\operatorname{im}\Gamma_T = \{\delta \in h_{\operatorname{Spec} R}(T) | \delta(I)T = T\}. \tag{19}$$

Let $f_1, \ldots, f_m \in R^{lm}$ denote a basis for $\Psi(\mathrm{id}_R)$, and suppose f_j has ith coordinate f_{ij}. Let a denote the $m \times m$-matrix whose pth column is $(f_{(i-1)m+1,p}, \ldots, f_{im,p})^t$, and let $I = \langle \det a \rangle$. We prove that I satisfies (19). That is, we prove that a homomorphism $\delta : R \to T$ has the property that

$$\Psi(\delta) = \langle (\beta(x_1), \ldots, \beta(x_{(i-1)m}), 1, 0, \ldots, 0, \beta(x_{(i-1)m+1}), \ldots, \beta(x_{lm-m})) \rangle \tag{20}$$

for some $\beta : B \to T$ iff $\delta(I)T = T$.

Since I is principle, $\delta : R \to T$ is such that $\delta(I)T = T$ iff $\delta(\det a)$ is a unit in T, which occurs iff the $m \times m$-matrix whose pth column is $(\delta(f_{(i-1)m+1,p}), \ldots, \delta(f_{im,p}))^t$ is invertible. By naturality of Ψ, $\Psi(\delta)$ is the image of the composition

$$T \otimes_R \Psi(\mathrm{id}_R) \to T \otimes_R R^n \to T^n$$

whose left arrow is induced by inclusion $\Psi(\mathrm{id}_R) \to R^n$. Thus, if $\delta(f_j)$ denotes $(\delta(f_{1j}), \ldots, \delta(f_{lm,j}))^t \in T^{lm}$, then $\delta(f_1), \ldots, \delta(f_m)$ is a basis for $\Psi(\delta)$. This implies that the $m \times m$-matrix whose pth column is $(\delta(f_{(i-1)m+1,p}), \ldots, \delta(f_{im,p}))^t$ is invertible iff the projection of $\Psi(\delta)$ to the $(i-1)m+1$ through the imth factors is onto. This occurs iff

$$N = \langle (\beta(x_1), \ldots, \beta(x_{(i-1)m}), 1, 0, \ldots, 0, \beta(x_{(i-1)m+1}), \ldots, \beta(x_{lm-m})) \rangle \subset \Psi(\delta)$$

for some $\beta : B \to T$. By Lemma 4.1, N is a free rank m summand of R^{ml}. We claim $N = \Psi(\delta)$. For, if $\mathfrak{m}$ is a maximal ideal of T, it follows from Nakayama's Lemma [1, Corollary 4.8, p. 124] that $N_{\mathfrak{m}} = \Psi(\delta)_{\mathfrak{m}}$. Hence, $N = \Psi(\delta)$, and the first assertion follows. To prove the second assertion, we note that the previous argument holds, mutatis mutandis, after replacing $F_A(m,n)$ by $H_A(m,n)$. $\square$

Corollary 4.7. *The open subfunctors Φ_i of $F_A(m,n)$ cover the K-rational points of $F_A(m,n)$. That is,*

$$F_A(m,n)(K) = \bigcup_i \Phi_i(h_{\operatorname{Spec} B}(K)).$$

Furthermore, if K is infinite and A is commutative, the open subfunctors Φ_i of $H_A(m,n)$ cover the K-rational points of $H_A(m,n)$.

Proof. By Lemma 4.6, the functors $\Phi_i : h_{\operatorname{Spec} B} \to F_A(m,n)$ are open. If M is a free rank m summand of K^{ml} which is A-invariant, then there exists an i such that some element of M has nonzero projection onto the $(i-1)m+1$ through the imth coordinates. Hence, since S is simple, M contains the submodule

$$\langle (b_1, \dots, b_{(i-1)m}, 1, 0, \dots, 0, b_{(i-1)m+1}, \dots, b_{lm-m}) \rangle \subset K^{lm},$$

where $b_1, \dots, b_{lm-m} \in K$. Thus, by Lemma 4.1,

$$M = \langle (b_1, \dots, b_{(i-1)m}, 1, 0, \dots, 0, b_{(i-1)m+1}, \dots, b_{lm-m}) \rangle,$$

so that $M \in \Phi_i(h_{\operatorname{Spec} B}(K))$. To prove the second assertion, we note that when K is infinite and A is commutative, Lemma 4.6 implies that Φ_i is an open subfunctor of $H_A(m,n)$. Thus, the second assertion follows from the first assertion and Remark 2.11. $\square$

We will use the following lemma to prove Theorem 4.9. The proof of the lemma is straightforward, so we omit it.

Lemma 4.8. *Let X be a topological space with open cover $\{A_i\}_{i \in I}$. If A_i is irreducible for all i and $A_i \cap A_j$ is nonempty for all $i, j \in I$, then X is irreducible.*

Theorem 4.9. *Let $\mathbb{F}$ denote the open subscheme of $\mathbb{F}_A(m,n)$ obtained by glueing the open subschemes of $\mathbb{F}_A(m,n)$ defined by Φ_i for $1 \leq i \leq l$. Then $\mathbb{F}$ is smooth, reduced, irreducible, of dimension $lm-m$ and has the same K-rational points as $\mathbb{F}_A(m,n)$. If K is infinite and A is commutative, $\mathbb{H}_A(m,n)$ contains a smooth, reduced, irreducible, open subscheme of dimension $lm-m$ which has the same K-rational points as $\mathbb{H}_A(m,n)$.*

Proof. The fact that $\mathbb{A}_K^{lm-m}$ is smooth, reduced and has dimension $lm-m$ implies that $\mathbb{F}$ is smooth, reduced and has dimension $lm-m$. The fact that $\mathbb{F}_A(m,n)$ and $\mathbb{F}$ have the same K-rational points follows from Corollary 4.7.

We denote the topological space of the open subscheme of $\mathbb{F}$ corresponding to Φ_i by $\mathbb{A}_i$. For any i, j, the set $\mathbb{A}_i \cap \mathbb{A}_j$ is nonempty. For example, if $i < j$, the intersection contains the point $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{A}_i$, where the nonzero entry occurs in the $(j-2)m+1$ position. The fact that $\mathbb{F}$ is irreducible now follows from Lemma 4.8. The proof of the second assertion is similar, and we omit it. $\square$

5. Two-sided vector spaces

In this section, we describe our notation and conventions regarding two-sided vector spaces, and we define the notion of rank of a two-sided vector space. We end the section by reviewing facts about simple two-sided vector spaces which are employed in the sequel.

Let V be a two-sided vector space. That is, V is a k-central $K-K$-bimodule which is finite-dimensional as a left K-module. Right multiplication by $x \in K$ defines an endomorphism $\phi(x)$ of ${}_K V$, and the right action of K on V is via the k-algebra homomorphism $\phi : K \to \operatorname{End}({}_K V)$. This motivates the following definition.

Definition 5.1. Let $\phi : K \to M_n(K)$ be a nonzero homomorphism. We denote by K_ϕ^n the two-sided vector space of left dimension n, where the left action is the usual one and the right action is via ϕ; that is,

$$x \cdot (v_1, \dots, v_n) = (xv_1, \dots, xv_n), \quad (v_1, \dots, v_n) \cdot x = (v_1, \dots, v_n)\phi(x). \tag{21}$$

We shall always write scalars as acting to the left of elements of K_ϕ^n and matrices acting to the right.

If V is a two-sided vector space, we let $\dim_K V$ denote the dimension of V as a left K-module. If $\dim_K V = n$, then choosing a left basis for V shows that $V \cong K^n_\phi$ for some homomorphism $\phi : K \to M_n(K)$. Throughout the rest of this paper, V will denote the two-sided vector space K^n_ϕ, W will denote a two-sided vector space and S will denote a simple two-sided vector space.

We denote the category of two-sided vector spaces by $\mathsf{Vect}K$. We shall denote by $K^B_i(K)$ the Quillen K-theory of $\mathsf{Vect}(K)$ (the superscript stands for "bimodule"). The groups $K^B_i(B)$ were computed in [4, Theorem 4.1].

Definition 5.2. The rank of a two-sided vector space W, denoted $[W]$, is the class of W in $K^B_0(K)$.

Thus, the rank of W is just the sums of the ranks of the simples (with multiplicity) appearing in the composition series of W.

We conclude this section with a description of the simple objects in $\mathsf{Vect}K$. Let $\overline{K}$ denote an algebraic closure of K. We write $\mathrm{Emb}(K)$ for the set of k-linear embeddings of K into $\bar{K}$, and $G = G(K)$ for the group $\mathrm{Aut}(\bar{K}/K)$.

The group G acts on $\mathrm{Emb}(K)$ by left composition. Given $\lambda \in \mathrm{Emb}(K)$, we denote the orbit of λ under this action by λ^G, and we write $K(\lambda)$ for the composite field $K \vee \mathrm{im}(\lambda)$.

We denote the set of finite orbits of $\mathrm{Emb}(K)$ under the action of G by $\Lambda(K)$. The following is a consequence of the proof of [4, Theorem 3.2]:

Theorem 5.3. *If K is perfect, there is a bijection from simple objects in $\mathsf{Vect}(K)$ to $\Lambda(K)$. Moreover, if V is a simple two-sided vector space mapping to $\lambda^G \in \Lambda(K)$, and if $\lambda^G = \{\sigma_1\lambda, \ldots, \sigma_m\lambda\}$, then $\dim_K V = |\lambda^G|$ and there is a basis for the image of the composition*

$$K(\lambda) \otimes_K V \xrightarrow{=} K(\lambda) \otimes_K K^n \xrightarrow{\cong} K(\lambda)^n$$

in which ϕ is a diagonal matrix with entries $\sigma_1\lambda, \ldots, \sigma_m\lambda$.

We denote the simple two-sided vector space corresponding to λ^G under the bijection in Theorem 5.3 by $V(\lambda)$.

We will need the following Corollary to [6, Lemma 3.13]:

Lemma 5.4. *Let F denote an extension field of k. If S and S' are left finite-dimensional, non-isomorphic simple $F \otimes_k K$-modules, then $\mathrm{Ext}^1_{F\otimes_k K}(S, S') = 0$.*

Since a two-sided vector space is just a $K \otimes_k K$-module, Lemma 5.4 implies that $V \cong V_1 \oplus \cdots \oplus V_r$, where V_i is S_i-homogeneous for some simple S_i.

6. Parameter spaces of two-sided subspaces of V

The purpose of this section is to use $\mathbb{F}_A(m, n)$, $\mathbb{G}_A(m, n)$, and $\mathbb{H}_A(m, n)$ to construct and study parameter spaces of two-sided subspaces of V.

6.1. The functors $F_\phi([W], V)$, $G_\phi([W], V)$, and $H_\phi([W], V)$.

Definition 6.1. If V is S-homogeneous and W is a two-sided vector space of rank $q[S]$, we let $F_\phi([W], V)(-) : K - \mathsf{alg} \to \mathsf{Sets}$ denote the functor $F_{\mathrm{im}\,\phi}(qm, n)$.

If W is not S-homogeneous, we let $F_\phi([W], V)(R) = \emptyset$.

Now suppose $V = V_1 \oplus \cdots \oplus V_r$, where V_i is S_i-homogeneous and S_i is simple, $\phi_i(x)$ is the restriction of $\phi(x)$ to V_i, and $[W] = q_1[S_1] + \cdots + q_r[S_r]$. We let $F_\phi([W], V)(-) : K - \mathsf{alg} \to \mathsf{Sets}$ denote the functor

$$F_{\phi_1}(q_1[S_1], V_1) \times \cdots \times F_{\phi_r}(q_r[S_r], V_r),$$

where the product is taken over $h_{\operatorname{Spec} K}$.

If W has a composition factor not in $\{S_1, \ldots, S_r\}$, we let $F_\phi([W], V)(R) = \emptyset$.

We call elements of $F_\phi([W], V)(R)$ *free rank* $[W]$ *ϕ-invariant families over* $\operatorname{Spec} R$, or *free ϕ-invariant families* when W and R are understood.

Definition 6.2. If V is S-homogeneous and W is a two-sided vector space of rank $q[S]$, we let $G_\phi([W], V)(-) : K - \mathsf{alg} \to \mathsf{Sets}$ denote the functor $G_{\operatorname{im} \phi}(qm, n)$.

If W is not S-homogeneous, we let $G_\phi([W], V)(R) = \emptyset$.

Now suppose $V = V_1 \oplus \cdots \oplus V_r$, where V_i is S_i-homogeneous and S_i is simple, $\phi_i(x)$ is the restriction of $\phi(x)$ to V_i, and $[W] = q_1[S_1] + \cdots + q_r[S_r]$. We let $G_\phi([W], V)(-) : K - \mathsf{alg} \to \mathsf{Sets}$ denote the functor

$$G_{\phi_1}(q_1[S_1], V_1) \times \cdots \times G_{\phi_r}(q_r[S_r], V_r),$$

where the product is taken over $h_{\operatorname{Spec} K}$.

If W has a composition factor not in $\{S_1, \ldots, S_r\}$, we let $G_\phi([W], V)(R) = \emptyset$.

We call elements of $G_\phi([W], V)(R)$ *free rank* $[W]$ *families generated by ϕ-invariants over* $\operatorname{Spec} R$, or *free families generated by ϕ-invariants* when W and R are understood.

Definition 6.3. If V is S-homogeneous and W is a two-sided vector space of rank $q[S]$, we let $H_\phi([W], V)(-) : K - \mathsf{alg} \to \mathsf{Sets}$ denote the functor $H_{\operatorname{im} \phi}(qm, n)$.

If W is not S-homogeneous, we let $H_\phi([W], V)(R) = \emptyset$.

Now suppose $V = V_1 \oplus \cdots \oplus V_r$, where V_i is S_i-homogeneous and S_i is simple, $\phi_i(x)$ is the restriction of $\phi(x)$ to V_i, and $[W] = q_1[S_1] + \cdots + q_r[S_r]$. We let $H_\phi([W], V)(-) : K - \mathsf{alg} \to \mathsf{Sets}$ denote the functor

$$H_{\phi_1}(q_1[S_1], V_1) \times \cdots \times H_{\phi_r}(q_r[S_r], V_r),$$

where the product is taken over $h_{\operatorname{Spec} K}$.

If W has a composition factor not in $\{S_1, \ldots, S_r\}$, we let $H_\phi([W], V)(R) = \emptyset$.

We call elements of $H_\phi([W], V)(R)$ *free rank* $[W]$ *ϕ-invariant families generated by ϕ-invariants over* $\operatorname{Spec} R$, or *free ϕ-invariant families generated by ϕ-invariants* when W and R are understood.

Lemma 6.4. *The K-rational points of $F_\phi([W], V)$, $G_\phi([W], V)$, and $H_\phi([W], V)$ are equal to the set of two-sided rank $[W]$ subspaces of V.*

Proof. We first show that the three functors above have the same K-rational points. From the definitions of $F_\phi([W], V)$, $G_\phi([W], V)$, and $H_\phi([W], V)$, it suffices to prove the result when V is homogeneous. Thus, it suffices to prove

$$F_{\operatorname{im} \phi}(m, n)(K) = G_{\operatorname{im} \phi}(m, n)(K) = H_{\operatorname{im} \phi}(m, n)(K)$$

when K^n is homogeneous as a $K \otimes_k \operatorname{im} \phi$-module. Since $\phi : K \to M_n(K)$ is a ring homomorphism, $\operatorname{im} \phi - \{0\} \subset GL_n(K)$. Thus, the assertion follows from Remark 2.9 and Remark 2.11.

To complete the proof of the lemma, it suffices to prove that

$$F_{\phi}([W],V)(K) = \{\text{two-sided rank } [W] \text{ subspaces of } V\}.$$

If V is homogeneous, this follows immediately from the definition of $F_{\mathrm{im}\,\phi}(m,n)$. If V is not homogeneous, the result follows from the fact that V, and any two-sided subspace of V, has a direct sum decomposition into its homogeneous components. $\square$

We now find conditions under which $F_{\phi}([S],V) \neq G_{\phi}([S],V)$ and $F_{\phi}([S],V) \neq H_{\phi}([S],V)$.

Lemma 6.5. *Suppose $\lambda_1,\ldots,\lambda_m \in \mathrm{Emb}(K)$ are distinct and $|k| > m$. If*

$$\{i_1,\ldots,i_m\} \subset \{1,\ldots,m\}$$

is a multiset with repetitions, then there exists an $a \in K$ such that

$$\prod_{j=1}^{m} \lambda_j(a) \neq \prod_{i=1}^{m} \lambda_{i_j}(a). \tag{22}$$

Proof. First we claim that there exists an element $b \in K$ such that there is an inequality of multisets

$$\{\lambda_1(b),\ldots,\lambda_m(b)\} \neq \{\lambda_{i_1}(b),\ldots,\lambda_{i_m}(b)\}.$$

If not, we would have

$$\sum_{j=1}^{m} \lambda_j = \sum_{j=1}^{m} \lambda_{i_j},$$

which is a nontrivial dependency relation among $\{\lambda_1,\ldots,\lambda_m\}$ as k-linear functions from K to $\overline{K}$. This contradicts the linear independence of characters from K to $\overline{K}$, which establishes our claim.

With b as above, we have

$$\prod_{j=1}^{m}(x-\lambda_j(b)) \neq \prod_{j=1}^{m}(x-\lambda_{i_j}(b))$$

in the ring $\overline{K}[x]$. Thus

$$f(x) = \prod_{j=1}^{m}(x-\lambda_j(b)) - \prod_{j=1}^{m}(x-\lambda_{i_j}(b))$$

has at most m roots. Since $|k| > m$, we may choose $c \in k$ such that $f(c) \neq 0$. Since λ_i is k-linear, we have

$$\prod_{j=1}^{m}(\lambda_j(c-b)) - \prod_{j=1}^{m}(\lambda_{i_j}(c-b)) = \prod_{j=1}^{m}(c-\lambda_j(b)) - \prod_{j=1}^{m}(c-\lambda_{i_j}(b)) = f(c) \neq 0.$$

Thus (22) holds with $a = c - b$. $\square$

Corollary 6.6. *Suppose $m \in \mathbb{N}$ is such that $|k| > m > 1$, K is perfect and $\lambda \in \mathrm{Emb}(K)$ is such that $|\lambda^G| = m$ (see Section 5 for notation). If $V(\lambda)^{\oplus 2} \subset V$, then there exist free rank $[V(\lambda)]$ ϕ-invariant families over* $\mathrm{Spec}\,K(\lambda)$ *which are not generated by* $\mathrm{im}\,\phi$*-invariants.*

Proof. Suppose $\lambda^G = \{\sigma_1\lambda, \ldots, \sigma_m\lambda\}$. Let M be a free rank $[V(\lambda)]$ ϕ-invariant family over $\operatorname{Spec} K(\lambda)$ which is generated by ϕ-eigenvectors $v_1, \ldots, v_m$, with eigenvalues

$$\sigma_{i_1}\lambda, \ldots, \sigma_{i_m}\lambda$$

respectively, such that the multiset $\{i_1, \ldots, i_m\}$ has repetitions. Then the eigenvalue of any generator of $\bigwedge^m M$ equals $\prod_j \sigma_{i_j}\lambda$. On the other hand, if M were generated by $\operatorname{im}\phi$-invariants, any generator of $\bigwedge^m M$ would have eigenvalue $\prod_j \sigma_j\lambda$. Thus, by Lemma 6.5, M is a free rank $[V(\lambda)]$ ϕ-invariant family which is not generated by $\operatorname{im}\phi$-invariants. □

Example 6.7. Suppose $\rho = \sqrt[3]{2}$, ζ is a primitive 3rd root of unity, $k = \mathbb{Q}$ and $K = \mathbb{Q}(\rho)$. For $i = 0, 1$, let

$$\lambda_i(\sum_{l=0}^{2} a_l\rho^l) = a_i\rho^i - a_2\rho^2$$

and let $\lambda(x) = \lambda_0(x) + \lambda_1(x)\zeta$. Then $V(\lambda)$ is a two-dimensional simple two-sided vector space [4, Example 3.9], and thus, by Corollary 6.6, $V = V(\lambda)^{\oplus 2}$ contains free rank $V(\lambda)$ ϕ-invariant families over $\operatorname{Spec} K(\lambda)$ which are not generated by $\operatorname{im}\phi$-invariants. In other words,

$$F_\phi([V(\lambda)], V(\lambda)^{\oplus 2})(K(\lambda)) \neq H_\phi([V(\lambda)], V(\lambda)^{\oplus 2})(K(\lambda)).$$

It follows immediately that

$$F_\phi([V(\lambda)], V(\lambda)^{\oplus 2})(K(\lambda)) \neq G_\phi([V(\lambda)], V(\lambda)^{\oplus 2})(K(\lambda)).$$

Remark 6.8. It follows from the previous example and the definitions of $F_\phi([W], V)$, $G_\phi([W], V)$, and $H_\phi([W], V)$ that there exist A, m and n such that $F_A(m, n) \neq G_A(m, n)$ and $F_A(m, n) \neq H_A(m, n)$.

6.2. **F-rational points of $G_\phi([S], V)$ and $H_\phi([S], V)$.** Let F be an extension field of K. In this subsection, we show that every element of $G_\phi([S], V)(F)$ and of $H_\phi([S], V)(F)$ is isomorphic to $F \otimes_K S$ as $F \otimes_k K$-modules. Throughout this subsection, we assume, without loss of generality, that V is S-homogeneous. Since, by Lemma 2.8, $G_\phi([S], V)(F) = H_\phi([S], V)(F)$, it suffices to prove the result for $G_\phi([S], V)(F)$. We assume throughout this subsection that $\dim_K S = m$.

Since the sum of all simple submodules of V is a direct summand of V as a left K-module, V has a left K-module decomposition

$$V = L \oplus N, \tag{23}$$

where N contains no simple two-sided subspaces of V and $L = S^{\oplus l}$ is a direct sum of simple two-sided subspaces of V.

Lemma 6.9. *Every free rank $[S]$ family generated by ϕ-invariants over* $\operatorname{Spec} F$ *is contained in $F \otimes_K L$.*

Proof. Assume $N \neq 0$ and suppose M is a free rank $[S]$ family generated by ϕ-invariants over $\operatorname{Spec} F$, with basis $\{v_i + w_i\}_{i=1}^m$, where v_i is an element of the image of the composition

$$F \otimes_K N \to F \otimes_K V \xrightarrow{\cong} F^n \tag{24}$$

induced by the inclusion $N \subset V$, and w_i is an element of the image of the composition

$$F \otimes_K L \to F \otimes_K V \overset{\cong}{\to} F^n \tag{25}$$

induced by the inclusion $L \subset V$. Since M is generated by $\operatorname{im} \phi$-invariants, $\wedge^m M$ is contained in the image of the composition

$$F \otimes_K \bigwedge^m L \to F \otimes_K \bigwedge^m V \to \bigwedge^m F^n$$

induced by the inclusion $\bigwedge^m L \to \bigwedge^m V$. On the other hand,

$$(v_1 + w_1) \wedge \cdots \wedge (v_m + w_m) = w_1 \wedge \cdots \wedge w_m + \text{wedges with at least one } v_i. \tag{26}$$

Since $\bigwedge^m M \neq 0$, this implies that $w_1 \wedge \cdots \wedge w_m \neq 0$, and that the sum of the other terms on the right-hand side of (26) equals 0. We claim each of the terms $v_1 \wedge w_2 \wedge \cdots \wedge w_m, \ldots, w_1 \wedge \cdots \wedge w_{m-1} \wedge v_m$ equals zero, which would prove the assertion. To this end, let $f_1, \ldots, f_p$ denote a basis for the image of (24), and suppose $f_{p+1}, \ldots, f_q, w_1, \ldots, w_m$ is a basis for the image of (25). Let

$$B_1 = \{w_1 \wedge \cdots \wedge w_m\} \cup \left(\bigcup_{1 \leq j_1 < \cdots < j_m \leq q} \{f_{j_1} \wedge \cdots \wedge f_{j_m}\} \right),$$

$$B_2 = \bigcup_{r=2}^{m-1} \left(\bigcup_{\substack{1 \leq i_1 < \cdots < i_r \leq q \\ 1 \leq i_{r+1} < \cdots < i_m \leq m}} \{f_{i_1} \wedge \cdots \wedge f_{i_r} \wedge w_{i_{r+1}} \wedge \cdots \wedge w_{i_m}\} \right),$$

$$B_3 = \left(\bigcup_{i=1}^{q} \{w_1 \wedge f_i \wedge w_3 \wedge \cdots \wedge w_m\} \right) \cup \cdots \cup \left(\bigcup_{i=1}^{q} \{w_1 \wedge \cdots \wedge w_{m-1} \wedge f_i\} \right),$$

and

$$B_4 = \bigcup_{i=1}^{q} \{f_i \wedge w_2 \wedge w_3 \wedge \cdots \wedge w_m\}.$$

The sets B_1, B_2, B_3, B_4 form a partition of a basis for $\bigwedge^m F^n$. Since the right-hand side of (26) equals $w_1 \wedge \cdots \wedge w_m$, $v_1 \wedge w_2 \wedge \cdots \wedge w_m$ is a linear combination of elements in $B_1 \cup B_2 \cup B_3$. On the other hand, $v_1 \wedge w_2 \wedge \cdots \wedge w_m$ is a linear combination of elements in B_4. We conclude that $v_1 \wedge w_2 \wedge \cdots \wedge w_m = 0$. A similar argument implies that $w_1 \wedge \cdots \wedge w_{i-1} \wedge v_i \wedge w_{i+1} \wedge \cdots \wedge w_m = 0$ for $1 \leq i \leq m$, and the result follows. □

Theorem 6.10. *Suppose $|k| > m$, K is perfect, and $K \subset F$ is an extension of fields. If M is a free rank $[S]$ family generated by ϕ-invariants over* Spec F*, then $M \cong F \otimes_K S$ as $F \otimes_k K$-modules.*

Proof. By Lemma 6.9, we may assume M is contained in the image of the composition

$$F \otimes_K L \to F \otimes_K V \overset{\cong}{\to} F^n$$

induced by the inclusion $L \subset V$. Thus, we may assume V is semisimple.

Let $\overline{F}$ denote an algebraic closure of F containing $\overline{K}$, let $\overline{M}$ denote the image of the composition

$$\overline{F} \otimes_F M \to \overline{F} \otimes_F F^n \overset{\cong}{\to} \overline{F}^n$$

whose left arrow is induced by inclusion, and let $\overline{M}$ have generators $w_1, \ldots, w_m$ as an $\overline{F}$-module. By Lemma 2.4, $\overline{M}$ is generated by $\operatorname{im} \phi$-invariants. Thus, Theorem 5.3 implies that the ϕ-eigenvalues of $w_1 \wedge \cdots \wedge w_m$ must equal $\sigma_1 \lambda(x) \cdots \sigma_m \lambda(x)$ for all

$x \in K$, where λ is a k-linear embedding of K into $\overline{K}$, $\sigma_1, \ldots, \sigma_m$ are automorphisms of $\overline{K}$ over K, and $\{\sigma_1\lambda, \ldots, \sigma_m\lambda\}$ are distinct. By Lemma 2.8, $\overline{M}$ is ϕ-invariant. Thus, $\overline{M}$ has a ϕ-eigenvector, v_1. Since v_1 is also an eigenvector in $\overline{F}^n \supset \overline{K}^n$, it must have eigenvalue $\sigma_{i_1}\lambda$. For $1 < j \leq m$, let $v_j \in \overline{M}$ be such that $v_j + \langle v_1, \ldots, v_{j-1}\rangle$ is a ϕ-eigenvector for $\overline{M}/\langle v_1, v_2, \ldots, v_{j-1}\rangle$, where $\langle v_1, \ldots, v_{j-1}\rangle$ denotes the $\overline{F} \otimes_k K$-module generated by $v_1, \ldots, v_{j-1}$. Then $v_j + \langle v_1, \ldots, v_{j-1}\rangle$ has eigenvalue $\sigma_{i_j}\lambda$, and, thus, in the basis $\{v_1, \ldots, v_m\}$, $\phi(x)|_{\overline{M}}$ is upper-triangular with diagonal entries $\sigma_{i_1}\lambda(x), \ldots, \sigma_{i_m}\lambda(x)$. Therefore,

$$\det \phi(x)|_{\overline{M}} = \sigma_{i_1}\lambda(x) \cdots \sigma_{i_m}\lambda(x).$$

By Lemma 6.5, we must have $\{i_1, \ldots, i_m\} = \{1, \ldots, m\}$. Since, by Lemma 5.4, extensions of distinct simple left finite-dimensional $\overline{F} \otimes_k K$-modules are split, there exists a basis for $\overline{M}$ such that $\phi(x)|_{\overline{M}}$ is diagonal with entries $\sigma_1\lambda(x), \ldots, \sigma_m\lambda(x)$. It follows that $\overline{F} \otimes_F M \cong \overline{F} \otimes_F (F \otimes_K S)$ as $\overline{F} \otimes_k K$-modules. Thus, by an argument similar to that given in the proof of [4, Lemma 2.4], we conclude that $M \cong F \otimes_K S$ as $F \otimes_k K$-modules. ☐

6.3. **The geometry of $\mathbb{F}_\phi([W], V)$, $\mathbb{G}_\phi([W], V)$, and $\mathbb{H}_\phi([W], V)$.** For the readers convenience, we collect here some consequences of our study of the geometry of $F_A(m, n)$, $G_A(m, n)$ and $H_A(m, n)$ in the case that $A = \operatorname{im} \phi$, where $\phi : K \to M_n(K)$ is a k-central ring homomorphism. We assume throughout the remainder of this section that $[V] = l_1[S_1] + \cdots + l_r[S_r]$, where $S_1, \ldots, S_r$ are non-isomorphic simple modules with $\dim S_i = m_i$, and that $\phi_i(x)$ is the restriction of $\phi(x)$ to the S_i-homogeneous summand of V. Finally, we assume all products of schemes are over $\operatorname{Spec} K$.

Theorem 6.11. *The functors $F_\phi(q_1[S_1]+\cdots+q_r[S_r], V)$, $G_\phi(q_1[S_1]+\cdots+q_r[S_r], V)$, and $H_\phi(q_1[S_1] + \cdots + q_r[S_r], V)$ are represented by*

$$\prod_{i=1}^{r} \mathbb{F}_{\operatorname{im} \phi_i}(m_i q_i, m_i l_i), \prod_{i=1}^{r} \mathbb{G}_{\operatorname{im} \phi_i}(m_i q_i, m_i l_i), \text{ and } \prod_{i=1}^{r} \mathbb{H}_{\operatorname{im} \phi_i}(m_i q_i, m_i l_i)$$

respectively.

Proof. Since $F_A(m, n)$, $G_A(m, n)$, and $H_A(m, n)$ are representable by $\mathbb{F}_A(m, n)$, $\mathbb{G}_A(m, n)$, and $\mathbb{H}_A(m, n)$, the result follows from [2, p. 260]. ☐

We denote the schemes representing $F_\phi([W], V)$, $G_\phi([W], V)$, and $H_\phi([W], V)$ by $\mathbb{F}_\phi([W], V)$, $\mathbb{G}_\phi([W], V)$, and $\mathbb{H}_\phi([W], V)$, respectively.

Corollary 6.12. *If K/k is finite and Galois then $F_\phi([W], V) = G_\phi([W], V) = H_\phi([W], V)$ and $\mathbb{F}_\phi(q_1[S_1] + \cdots + q_r[S_r], V)$ equals*

$$\prod_{i=1}^{r} \mathbb{G}(q_i, l_i).$$

Proof. We prove that $\mathbb{F}_\phi(q_1[S_1]+\cdots+q_r[S_r], V) = \prod_{i=1}^{r} \mathbb{G}(q_i, l_i)$. The other assertions follow similarly. By the previous result, it suffices to prove that $\mathbb{F}_{\operatorname{im} \phi_i}(m_i q_i, m_i l_i) = \mathbb{G}(q_i, l_i)$. The hypothesis on K/k is equivalent to K being a finite, separable extension of k such that $\operatorname{Aut} K = \operatorname{Emb} K$. Thus, $m_i = 1$ [4, Theorem 3.2] and $S_i \cong K_{\sigma_i}$ for some k-linear automorphism σ_i of K (note that, in this case, we do not require that K be perfect to apply [4, Theorem 3.2]). Since $K \otimes_k K$ is semisimple, ϕ_i is a diagonal matrix with each diagonal entry equal to σ_i, and the assertion follows. □

Remark 6.13. The previous result also follows from the second part of [5, Theorem 1, p. 321].

Now assume that V is semisimple. Before we state our next result, we need to introduce some notation. For $1 \leq i \leq r$, let $B_i = K[x_{i,1}, \ldots, x_{i,l_i m_i - m_i}]$, let $B = K[\{x_{i,1}, \ldots, x_{i,l_i m_i - m_i}\}_{i=1}^r]$ and, for each r-tuple $J = (j_1, \ldots, j_r)$ such that $1 \leq j_i \leq l_i$ define an inclusion of functors

$$\Phi_J : h_{\operatorname{Spec} B_1} \times \cdots \times h_{\operatorname{Spec} B_r} \to F_{\phi_1}([S_1], V_1) \times \cdots \times F_{\phi_r}([S_r], V_r)$$

by $\Phi_J = \Phi_{j_1} \times \cdots \times \Phi_{j_r}$, where Φ_i is defined by (18), and where all products are over $h_{\operatorname{Spec} K}$. We abuse notation by letting Φ_J denote the induced natural transformation

$$h_{\operatorname{Spec} B} \overset{\cong}{\to} h_{\operatorname{Spec} B_1} \times \cdots \times h_{\operatorname{Spec} B_r} \overset{\Phi_J}{\to} F_\phi([S_1] + \cdots + [S_r], V).$$

In a similar fashion, we can define an inclusion of functors

$$\Phi_J : h_{\operatorname{Spec} B_1} \times \cdots \times h_{\operatorname{Spec} B_r} \to H_{\phi_1}([S_1], V_1) \times \cdots \times H_{\phi_r}([S_r], V_r),$$

where we have abused notation as in Section 4.

The following result is an immediate consequence of Lemma 4.6 and Corollary 4.7.

Theorem 6.14. *For all r-tuples $J = (j_1, \ldots, j_r)$ such that $1 \leq j_i \leq l_i$, $\Phi_J : h_{\operatorname{Spec} B} \to F_\phi([S_1]+\cdots+[S_r], V)$ is an open subfunctor, and the open subfunctors Φ_J cover the K-rational points of $F_\phi([S_1]+\cdots+[S_r], V)$. Furthermore, if K is infinite, the same result holds for $H_\phi([S_1]+\cdots+[S_r], V)$ in place of $F_\phi([S_1]+\cdots+[S_r], V)$.*

The following follows from the above result and from an argument similar to that used to prove Theorem 4.9.

Corollary 6.15. *$\mathbb{F}_\phi([S_1]+\cdots+[S_r], V)$ and $\mathbb{H}_\phi([S_1]+\cdots+[S_r], V)$ contain smooth, reduced, irreducible open subschemes of dimension $\sum_{i=1}^r l_i m_i - m_i$ which cover their K-rational points.*

The following example illustrates the fact that the open subfunctors Φ_J do not always form an open cover of $F_\phi([S], S^{\oplus l})$ or $H_\phi([S], S^{\oplus l})$.

Example 6.16. Suppose $\rho = \sqrt[3]{2}$, ζ is a primitive 3rd root of unity, $k = \mathbb{Q}$ and $K = \mathbb{Q}(\rho)$. For $i = 0, 1$, let

$$\lambda_i(\sum_{l=0}^{2} a_l \rho^l) = a_i \rho^i - a_2 \rho^2$$

and let $\lambda(x) = \lambda_0(x) + \lambda_1(x)\zeta$. Let $V(\lambda)$ denote the corresponding two-dimensional simple $K \otimes_k K$-module, so that the right action of K on $V(\lambda)$ is given by $\phi(x) = \begin{pmatrix} \lambda_0(x) & -\lambda_1(x) \\ \lambda_1(x) & -\lambda_1(x) + \lambda_0(x) \end{pmatrix}$ [4, Example 3.9]. Let $V = V(\lambda)^{\oplus 2}$, and let $\{e_i\}_{i=1}^4$ denote the standard unit vectors of $K(\zeta)^4$. Then

$$M = \operatorname{Span}_{K(\zeta)}\{e_1 + \zeta e_2, e_3 + \zeta^2 e_4\} \subset K(\zeta)^4 \cong K(\zeta) \otimes_K V$$

is a free ϕ-invariant rank $[V(\lambda)]$ family over $\operatorname{Spec} K(\zeta)$ whose projections onto the first and second coordinates of $K(\zeta)^4$, and onto the third and fourth coordinates of

$K(\zeta)^4$, are not onto. In particular, M is not an element of $\bigcup_{J=1}^{2} \Phi_J(h_{\operatorname{Spec} A}(K(\zeta)))$, and hence by [2, Exercise VI-II, p. 256], the open subfunctors Φ_J do not cover $F_\phi([V(\lambda)], V(\lambda)^{\oplus 2})$.

We claim that M is generated by $\operatorname{im}\phi$-invariants, which would establish that the open subfunctors Φ_J do not cover $H_\phi([V(\lambda)], V(\lambda)^{\oplus 2})$. To prove the claim, we first note that

$$(e_1 + \zeta e_2) \wedge (e_3 + \zeta^2 e_4) = e_1 \wedge e_3 + \zeta^2 e_1 \wedge e_4 + \zeta e_2 \wedge e_3 + e_2 \wedge e_4. \tag{27}$$

On the other hand, if we let $w_1 = e_1$, $w_2 = e_3$, $a_1 = 1$ and $a_2 = \rho$, then

$$w_1\phi(a_1) \wedge w_2\phi(a_2) + w_2\phi(a_1) \wedge w_1\phi(a_2) = -\rho(e_1 \wedge e_4 - e_2 \wedge e_3)$$

is an element of $\bigwedge^2_{\operatorname{im}\phi}$ by Proposition 4.3. Similarly, if we let $w_1 = e_1$, $w_2 = e_4$, $a_1 = 1$ and $a_2 = \rho$, then

$$w_1\phi(a_1) \wedge w_2\phi(a_2) + w_2\phi(a_1) \wedge w_1\phi(a_2) = \rho(e_1 \wedge e_3 - e_1 \wedge e_4 - e_4 \wedge e_2)$$

is an element of $\bigwedge^2_{\operatorname{im}\phi}$ by Proposition 4.3. It follows that (27) is an element of the image of $K(\zeta) \otimes_K \bigwedge^2_{\operatorname{im}\phi} \to K(\zeta) \otimes_K \bigwedge^2 K^4 \overset{\cong}{\to} \bigwedge^2 K(\zeta)^4$, and hence that M is generated by $\operatorname{im}\phi$-invariants.

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