Classical Integral Transforms in Semi-commutative Algebraic Geometry

Adam Nyman

Western Washington University

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Conventions and Notation

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- "scheme"=commutative k-scheme
- "bimodule"=object in $\mathsf{Bimod}_k(\mathsf{C}, \mathsf{D})$

Part 1

Integral Transforms and Bimod: Examples

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$$\mathrm{Mod}(R^{op} \otimes_k S) \to \mathrm{Bimod}_k(\mathrm{Mod} R, \mathrm{Mod} S).$$

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as left adjoint to

$$\text{Hom}_{\mathsf{A}}(\mathcal{F}, -) : \mathsf{A} \to \text{Mod}R$$

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Theorem (Artin-Zhang 1994)

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Theorem (Artin-Zhang 1994)

If $F : \text{mod}\mathcal{A} \longrightarrow \text{mod}\mathcal{B}$ is an equivalence then

$$F \cong \pi_*(\tau^*(-) \otimes_{\tau^*\mathcal{A}} \mathcal{F})$$

for some $\mathcal{F} \in \text{mod}(\mathcal{A}^{op} \otimes_k \mathcal{B})$.

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"Classical"=not between derived or dg or ∞-categories

Part 2

Non- and Semi-commutative Algebraic Geometry

Non-commutative Algebraic Geometry: Noncommutative Spaces

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- Proj $A :=$ GrA/Tors$A$ where A is $\mathbb{Z}$-graded

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If $f : Y \to X$ is a morphism of commutative schemes, (f^*, f_*) is an adjoint pair.

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If $f : Y \to X$ is a morphism of commutative schemes, (f^*, f_*) is an adjoint pair.

Adjoint functor theorem $\Rightarrow$

$$\text{Morphisms } f : Y \to Z \quad \leftrightarrow \quad \text{Bimod}_k(\text{Mod}\,Z, \text{Mod}\,Y).$$

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e.g.

Let $f : Y \to X$ denote a morphism of schemes such that $(f^*, f_*, f^!)$ is an adjoint triple (e.g. a closed immersion of varieties).

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are morphisms of noncommutative spaces $Y \to X$ and $X \to Y$.

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- $- \otimes_{\mathcal{A}} \mathcal{F}$ denotes tensoring a right $\mathcal{A}$-module with $\mathcal{F}$ and

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$$\pi_*(- \otimes_{\mathcal{A}} \mathcal{F}) : \mathsf{Mod}\mathcal{A} \to \mathsf{Mod}Y$$

where

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Main Idea (Artin-Zhang 2001)

Constructions from commutative algebraic geometry which are local only over X exist in semi-commutative setting.

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an isomorphism
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e.g.

$X = \mathrm{Spec}\, k$, $\mathcal{A} \leftrightarrow R \Rightarrow {}_{\mathcal{A}}\mathrm{Mod}\, Y \equiv {}_{R}\mathrm{Mod}\, Y$

Integral Transforms in Semi-commutative Algebraic Geometry: Tensor products with Left $\mathcal{A}$-objects

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$$\mathcal{F} \text{ is } \textbf{flat}/\mathcal{A} \Leftrightarrow - \otimes_{\mathcal{A}} \mathcal{F} \text{ is exact.}$$

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- $\text{R}^i \pi_* \mathcal{N} := \text{H}^i(C^\cdot(\mathfrak{U}, \mathcal{N}))$

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3. When is $F \in \mathsf{Bimod}_k(\mathsf{Mod}\mathcal{A}, \mathsf{Mod}Y)$ an integral transform? If it isn't, how close is it to being an integral transform?

Part 4

Semi-commutative Algebraic Geometry: Maps from n.c. spaces to curves (w/D. Chan)

Throughout Part 4, $k = \overline{k}$

Commutative example

X =smooth curve

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Problem

To what extent do 1-3 hold in the semi-commutative setting?

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Semi-commutative Algebraic Geometry: Examples of n.c. smooth proper d-folds

e.g.

$$A = Skl(a,b,c) := \frac{k\langle x_0, x_1, x_2\rangle}{(ax_i x_{i+1} + bx_{i+1}x_i + cx_{i+2}^2 : i = 1,2,3 \mod 3)}$$

for generic $(a : b : c) \in \mathbb{P}^2$.

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Proj$A$ is a n.c. smooth proper 3-fold. Given $t \in Z(A)_2$, Proj$A/(t)$ is a n.c. smooth proper 2-fold.

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Remark: This specializes to the intersection product for curves on a comm. surface.

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For $P \in \operatorname{mod} Y$, there exists a Hilbert scheme Hilb P parameterizing quotients of P. Hilb P is countable union of projective schemes which is locally of finite type.

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Remark: In comm. case, 1-2 $\Rightarrow$ F is struct. sheaf of K-negative, rational curve with self-intersection 0. 3 is substitute for K-negativity, since we don't know $K.C < 0 \Rightarrow H^0(\mathcal{O}_C \otimes \omega) = 0$.

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Remark: In comm. case, 1-2 $\Rightarrow$ F is struct. sheaf of K-negative, rational curve with self-intersection 0. 3 is substitute for K-negativity, since we don't know $K.C < 0 \Rightarrow H^0(\mathcal{O}_C \otimes \omega) = 0$.

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Maps from n.c. spaces to curves (w/D. Chan): K-non-effective rational curves

Y = n.c. smooth proper surface with structure sheaf $\mathcal{O}_Y$

Say $F \in \operatorname{mod} Y$ is K-**non-effective rational curve with self-intersection 0** if

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If $f : Y \to X$ is a comm. or n.c. ruled surface then the structure sheaf of fiber is K-non-effective rational curve with self-intersection 0.

Semi-commutative Algebraic Geometry: Maps from n.c. spaces to curves (w/D. Chan)

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If X = smooth curve, $f : Y \to X$ = n.c. ruled surface and F = struct. sheaf of fiber

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If $X =$ smooth curve, $f : Y \to X =$ n.c. ruled surface and $F =$ struct. sheaf of fiber then F satisfies 1,2 and $f^* \cong \pi_*(- \otimes_{\mathcal{O}_X} \mathcal{F})$.

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3. When is $F \in \text{Bimod}_k(\text{Mod}\mathcal{A}, \text{Mod}Y)$ an integral transform? If it isn't, how close is it to being an integral transform?

Part 5

Integral Transforms and Bimod revisited

Integral Transforms and Bimod revisited: The Functor $(-)^{\flat}$.

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The collection $F^\flat(U) = \mathcal{F}_U$ induces a functor

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1. Find relationship between F and $\pi_*(- \otimes_{\mathcal{A}} F^\flat)$
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Integral Transforms and Bimod revisited: Totally Global Functors

$F \in \text{Bimod}_k(\text{Mod}\mathcal{A}, \text{Mod}Y)$ such that $F^\flat = 0$ are **totally global**

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4. Test Problem: Classify noetherian preserving $F \in \text{Bimod}_k(\text{Qcoh}\mathbb{P}^1, \text{Mod}k)$ in case $k = \overline{k}$.

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Thank you for your attention!