

Classical Integral Transforms in Semi-commutative Algebraic Geometry

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Conventions and Notation

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- “scheme” = commutative k -scheme
- “bimodule” = object in $\text{Bimod}_k(C, D)$

Part 1

Integral Transforms and Bimod: Examples

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R, S rings, \mathcal{F} an $R - S$ -bimodule

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$$\text{Mod}(R^{op} \otimes_k S) \rightarrow \text{Bimod}_k(\text{Mod}R, \text{Mod}S).$$

Integral Transforms and Bimod: A Generalization of Example 1

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as left adjoint to

$$\text{Hom}_A(\mathcal{F}, -) : A \rightarrow \text{Mod} R$$

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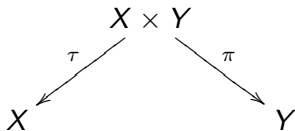
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Integral Transforms and Bimod: Examples

Example 2 Y scheme

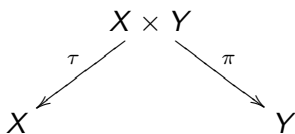
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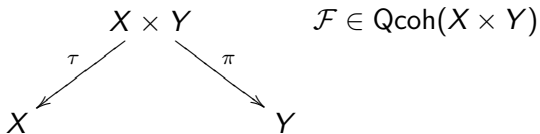
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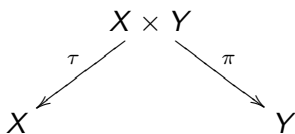
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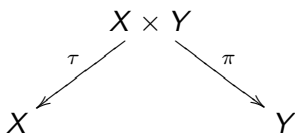
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If $f : Y \rightarrow X$ is a morphism of schemes then
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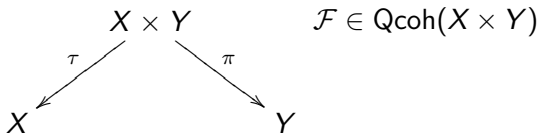
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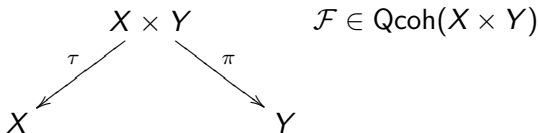
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Let $X = \mathbb{P}^1$ and $Y = \text{Spec } k$, $\mathcal{F} = \mathcal{O}_X$.

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Theorem (Artin-Zhang 1994)

If $F : \text{mod}\mathcal{A} \rightarrow \text{mod}\mathcal{B}$ is an equivalence then

$$F \cong \pi_* (\tau^* (-) \otimes_{\tau^* \mathcal{A}} \mathcal{F})$$

for some $\mathcal{F} \in \text{mod}(\mathcal{A}^{op} \otimes_k \mathcal{B})$.

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“Classical” = **not** between derived or dg or ∞ -categories

Part 2

Non- and Semi-commutative Algebraic Geometry

Non-commutative Algebraic Geometry: Noncommutative Spaces

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Motivation

If $f : Y \rightarrow X$ is a morphism of commutative schemes, (f^*, f_*) is an adjoint pair.

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Motivation

If $f : Y \rightarrow X$ is a morphism of commutative schemes, (f^*, f_*) is an adjoint pair.

Adjoint functor theorem \Rightarrow

$$\text{Morphisms } f : Y \rightarrow Z \quad \leftrightarrow \quad \text{Bimod}_k(\text{Mod } Z, \text{Mod } Y).$$

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Let $f : Y \rightarrow X$ denote a morphism of schemes such that $(f^*, f_*, f^!)$ is an adjoint triple (e.g. a closed immersion of varieties).

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The latter may not come from a morphism of schemes.

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Main Idea (Artin-Zhang 2001)

Constructions from commutative algebraic geometry which are local only over X exist in semi-commutative setting.

Integral Transforms in Semi-commutative Algebraic Geometry: Left \mathcal{A} -objects

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- Given $U \times V \xrightarrow{\text{pr}_1} U$ an isomorphism

$$\begin{array}{ccc} U \times V & \xrightarrow{\text{pr}_1} & U \\ \text{pr}_2 \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

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$$X = \text{Spec } k, \mathcal{A} \leftrightarrow R \Rightarrow {}_{\mathcal{A}}\text{Mod} Y \cong {}_R\text{Mod} Y$$

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- $R^i \pi_* \mathcal{N} := H^i(C^\bullet(\mathfrak{U}, \mathcal{N}))$

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Part 4

Semi-commutative Algebraic Geometry: Maps from n.c. spaces to curves (w/D. Chan)

Throughout Part 4, $k = \bar{k}$

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Commutative example

X = smooth curve

$f : Y \rightarrow X$ is comm. ruled surface with fiber C .

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Problem

To what extent do 1-3 hold in the semi-commutative setting?

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for generic $(a : b : c) \in \mathbb{P}^2$.

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Remark: This specializes to the intersection product for curves on a comm. surface.

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Semi-commutative Algebraic Geometry: Maps from n.c. spaces to curves (w/D. Chan)

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Semi-commutative Algebraic Geometry: Maps from n.c. spaces to curves (w/D. Chan)

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Semi-commutative Algebraic Geometry: Maps from n.c. spaces to curves (w/D. Chan)

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Part 5

Integral Transforms and Bimod revisited

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Theorem (Van den Bergh, Chan-N.)

The collection $F^b(U) = \mathcal{F}_U$ induces a functor

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Integral Transforms and Bimod revisited: Totally Global Functors

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- 4 Test Problem: Classify noetherian preserving $F \in \text{Bimod}_k(\text{Qcoh}\mathbb{P}^1, \text{Mod}k)$ in case $k = \bar{k}$.

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Claim B confirmed in case $\mathcal{A} = \mathcal{O}_X$ and $Y =$ scheme (N 2009)

Thank you for your attention!