SERRE FINITENESS AND SERRE VANISHING FOR NON-COMMUTATIVE $\mathbb{P}^1$-BUNDLES

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ABSTRACT. Suppose X is a smooth projective scheme of finite type over a field K, $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2, $\mathcal{A}$ is the non-commutative symmetric algebra generated by $\mathcal{E}$ and $\mathsf{Proj}\mathcal{A}$ is the corresponding non-commutative $\mathbb{P}^1$-bundle. We use the properties of the internal Hom functor $\underline{\mathcal{H}om}_{\mathsf{Gr}\mathcal{A}}(-,-)$ to prove versions of Serre finiteness and Serre vanishing for $\mathsf{Proj}\mathcal{A}$. As a corollary to Serre finiteness, we prove that $\mathsf{Proj}\mathcal{A}$ is Ext-finite. This fact is used in [2] to prove that if X is a smooth curve over Spec K, $\mathsf{Proj}\mathcal{A}$ has a Riemann-Roch theorem and an adjunction formula.

Keywords: non-commutative geometry, Serre finiteness, non-commutative projective bundle.

1. INTRODUCTION

Non-commutative $\mathbb{P}^1$-bundles over curves play a prominent role in the theory of non-commutative surfaces. For example, certain non-commutative quadrics are isomorphic to non-commutative $\mathbb{P}^1$-bundles over curves [9]. In addition, every non-commutative deformation of a Hirzebruch surface is given by a non-commutative $\mathbb{P}^1$-bundle over $\mathbb{P}^1$ [8, Theorem 7.4.1, p. 29].

The purpose of this paper is to prove versions of Serre finiteness and Serre vanishing (Theorem 3.5 (1) and (2), respectively) for non-commutative $\mathbb{P}^1$-bundles over smooth projective schemes of finite type over a field K. As a corollary to the first of these results, we prove that such non-commutative $\mathbb{P}^1$-bundles are Ext-finite. This fact is used to prove that non-commutative $\mathbb{P}^1$-bundles over smooth curves have a Riemann-Roch theorem and an adjunction formula [2].

We now review some important notions from non-commutative algebraic geometry in order to recall the definition of non-commutative $\mathbb{P}^1$-bundle. We conclude the introduction by relating the results of this paper to Mori's intersection theory.

If X is a quasi-compact and quasi-separated scheme, then $\mathsf{Mod}X$, the category of quasi-coherent sheaves on X, is a Grothendieck category. This leads to the following generalization of the notion of scheme, introduced by Van den Bergh in order to define a notion of blowing-up in the non-commutative setting.

Definition 1.1. [7] A **quasi-scheme** is a Grothendieck category $\mathsf{Mod}X$, which we denote by X. X is called a **noetherian** quasi-scheme if the category $\mathsf{Mod}X$ is locally noetherian. X is called a **quasi-scheme over K** if the category $\mathsf{Mod}X$ is K-linear.

If R is a ring and $\mathsf{Mod}R$ is the category of right R-modules, $\mathsf{Mod}R$ is a quasi-scheme, called the non-commutative affine scheme associated to R. If A is a graded

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ring, $\mathsf{Gr}A$ is the category of graded right A-modules, $\mathsf{Tors}A$ is the full subcategory of $\mathsf{Gr}A$ consisting of direct limits of right bounded modules, and $\mathsf{Proj}A$ is the quotient category $\mathsf{Gr}A/\mathsf{Tors}A$, then $\mathsf{Proj}A$ is a quasi-scheme called the non-commutative projective scheme associated to A. If A is an Artin-Schelter regular algebra of dimension 3 with the same hilbert series as a polynomial ring in 3 variables, $\mathsf{Proj}A$ is called a non-commutative $\mathbb{P}^2$.

The notion of non-commutative $\mathbb{P}^1$-bundle over a smooth scheme X generalizes that of commutative $\mathbb{P}^1$-bundle over X. In order to recall the definition of non-commutative $\mathbb{P}^1$-bundle, we review some preliminary notions. Let S be a scheme of finite type over $\operatorname{Spec} K$ and let X be an S-scheme. For $i = 1, 2$, let $\mathrm{pr}_i : X \times_S X \to X$ denote the standard projections, let $\delta : X \to X \times_S X$ denote the diagonal morphism, and let Δ denote the image of δ.

Definition 1.2. A **coherent $\mathcal{O}_X$-bimodule**, $\mathcal{E}$, is a coherent $\mathcal{O}_{X\times_S X}$-module such that $\mathrm{pr}_{i|\,\mathrm{Supp}\mathcal{E}}$ is finite for $i = 1, 2$. A coherent $\mathcal{O}_X$-bimodule $\mathcal{E}$ is **locally free of rank n** if $\mathrm{pr}_{i*}\,\mathcal{E}$ is locally free of rank n for $i = 1, 2$.

Now assume X is smooth. If $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule, then let $\mathcal{E}^*$ denote the dual of $\mathcal{E}$ [8, p. 6], and let $\mathcal{E}^{j*}$ denote the dual of $\mathcal{E}^{j-1*}$. Finally, let $\eta : \mathcal{O}_\Delta \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^*$ denote the counit from $\mathcal{O}_\Delta$ to the bimodule tensor product of $\mathcal{E}$ and $\mathcal{E}^*$ [8, p. 7].

Definition 1.3. [8, Section 4.1] Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-bimodule. The **non-commutative symmetric algebra generated by $\mathcal{E}$**, $\mathcal{A}$, is the sheaf-$\mathbb{Z}$-algebra generated by the $\mathcal{E}^{j*}$ subject to the relations $\eta(\mathcal{O}_\Delta)$.

A more explicit definition of non-commutative symmetric algebra is given in Section 2. We now recall the definition of non-commutative $\mathbb{P}^1$-bundle.

Definition 1.4. [8] Suppose X is a smooth scheme of finite type over K, $\mathcal{E}$ is a locally free $\mathcal{O}_X$-bimodule of rank 2 and $\mathcal{A}$ is the non-commutative symmetric algebra generated by $\mathcal{E}$. Let $\mathsf{Gr}\mathcal{A}$ denote the category of graded right $\mathcal{A}$-modules, let $\mathsf{Tors}\mathcal{A}$ denote the full subcategory of $\mathsf{Gr}\mathcal{A}$ consisting of direct limits of right-bounded modules, and let $\mathsf{Proj}\mathcal{A}$ denote the quotient of $\mathsf{Gr}\mathcal{A}$ by $\mathsf{Tors}\mathcal{A}$. The category $\mathsf{Proj}\mathcal{A}$ is a **non-commutative $\mathbb{P}^1$-bundle over X.**

This notion generalizes that of a commutative $\mathbb{P}^1$-bundle over X as follows. Let $\mathcal{E}$ be an $\mathcal{O}_X$-bimodule on which $\mathcal{O}_X$ acts centrally. Then $\mathcal{E}$ can be identified with the direct image $\mathrm{pr}_{i*}\mathcal{E}$ for $i = 1, 2$. If, furthermore, $\mathcal{E}$ is locally free of rank 2 and $\mathcal{A}$ is the non-commutative symmetric algebra generated by $\mathcal{E}$, Van den Bergh proves [8, Lemma 4.2.1] that the category $\mathsf{Proj}\mathcal{A}$ is equivalent to the category $\mathsf{Mod}\mathbb{P}_X(\mathrm{pr}_{i*}\,\mathcal{E})$, where $\mathbb{P}_X(-)$ is the usual (commutative) projectivization.

One of the major problems in non-commutative algebraic geometry is to classify non-commutative surfaces. Since intersection theory on commutative surfaces facilitates the classification of commutative surfaces, one expects intersection theory to be an important tool in non-commutative algebraic geometry. Mori shows [2, Theorem 3.11] that if Y is a noetherian quasi-scheme over a field K such that

1. Y is Ext-finite,
2. the cohomological dimension of Y is 2, and
3. Y satisfies Serre duality

then versions of the Riemann-Roch theorem and the adjunction formula hold for Y. Let X be a smooth curve over $\operatorname{Spec} K$. In [5], we prove that a non-commutative

$\mathbb{P}^1$-bundle over X satisfies (2) and (3) above (see Section 4 for a precise statement of these results). In this paper we prove that a non-commutative $\mathbb{P}^1$-bundle over a projective scheme of finite type satisfies (1) (Corollary 3.6). We conclude the paper by stating the versions of the Riemann-Roch theorem and the adjunction formula which hold for non-commutative $\mathbb{P}^1$-bundles.

In what follows, K is a field, X is a smooth, projective scheme of finite type over $\operatorname{Spec} K$, $\mathsf{Mod}X$ denotes the category of quasi-coherent $\mathcal{O}_X$-modules, and we abuse notation by calling objects in this category $\mathcal{O}_X$-modules.

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2. PRELIMINARIES

Before we prove Serre finiteness and Serre vanishing, we review the definition of non-commutative symmetric algebra and the definition and basic properties of the internal Hom functor $\underline{\mathcal{H}om}_{\mathsf{Gr}\mathcal{A}}(-,-)$ on $\mathsf{Gr}\mathcal{A}$.

Definition 2.1. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-bimodule. The **non-commutative symmetric algebra generated by $\mathcal{E}$** is the sheaf-$\mathbb{Z}$-algebra $\mathcal{A} = \underset{i,j\in\mathbb{Z}}{\oplus} \mathcal{A}_{ij}$ with components

- $\mathcal{A}_{ii} = \mathcal{O}_\Delta$
- $\mathcal{A}_{i,i+1} = \mathcal{E}^{i*}$,
- $\mathcal{A}_{ij} = \mathcal{A}_{i,i+1} \otimes \cdots \otimes \mathcal{A}_{j-1,j}/\mathcal{R}_{ij}$ for $j > i+1$, where $\mathcal{R}_{ij} \subset \mathcal{A}_{i,i+1} \otimes \cdots \otimes \mathcal{A}_{j-1,j}$ is the $\mathcal{O}_X$-bimodule
$$\sum_{k=i}^{j-2} \mathcal{A}_{i,i+1} \otimes \cdots \otimes \mathcal{A}_{k-1,k} \otimes \mathcal{Q}_k \otimes \mathcal{A}_{k+2,k+3} \otimes \cdots \otimes \mathcal{A}_{j-1,j},$$
and $\mathcal{Q}_i$ is the image of the unit map $\mathcal{O}_\Delta \to \mathcal{A}_{i,i+1} \otimes \mathcal{A}_{i+1,i+2}$, and
- $\mathcal{A}_{ij} = 0$ if $i > j$

and with multiplication, μ, defined as follows: for $i < j < k$,

$$\begin{aligned}\mathcal{A}_{ij} \otimes \mathcal{A}_{jk} &= \frac{\mathcal{A}_{i,i+1} \otimes \cdots \otimes \mathcal{A}_{j-1,j}}{\mathcal{R}_{ij}} \otimes \frac{\mathcal{A}_{j,j+1} \otimes \cdots \otimes \mathcal{A}_{k-1,k}}{\mathcal{R}_{jk}} \\ &\cong \frac{\mathcal{A}_{i,i+1} \otimes \cdots \otimes \mathcal{A}_{k-1,k}}{\mathcal{R}_{ij} \otimes \mathcal{A}_{j,j+1} \otimes \cdots \otimes \mathcal{A}_{k-1,k} + \mathcal{A}_{i,i+1} \otimes \cdots \otimes \mathcal{A}_{j-1,j} \otimes \mathcal{R}_{jk}}\end{aligned}$$

by [4, Corollary 3.18]. On the other hand,

$$\mathcal{R}_{ik} \cong \mathcal{R}_{ij} \otimes \mathcal{A}_{j,j+1} \otimes \cdots \otimes \mathcal{A}_{k-1,k} + \mathcal{A}_{i,i+1} \otimes \cdots \otimes \mathcal{A}_{j-1,j} \otimes \mathcal{R}_{jk} +$$
$$\mathcal{A}_{i,i+1} \otimes \cdots \otimes \mathcal{A}_{j-2,j-1} \otimes \mathcal{Q}_{j-1} \otimes \mathcal{A}_{j+1,j+2} \otimes \cdots \otimes \mathcal{A}_{k-1,k}.$$

Thus there is an epi $\mu_{ijk} : \mathcal{A}_{ij} \otimes \mathcal{A}_{jk} \to \mathcal{A}_{ik}$.

If $i = j$, let $\mu_{ijk} : \mathcal{A}_{ii} \otimes \mathcal{A}_{ik} \to \mathcal{A}_{ik}$ be the scalar multiplication map ${}_{\mathcal{O}}\mu : \mathcal{O}_\Delta \otimes \mathcal{A}_{ik} \to \mathcal{A}_{ik}$. Similarly, if $j = k$, let $\mu_{ijk} : \mathcal{A}_{ij} \otimes \mathcal{A}_{jj} \to \mathcal{A}_{ij}$ be the scalar multiplication map $\mu_{\mathcal{O}}$. Using the fact that the tensor product of bimodules is associative, one can check that multiplication is associative.

Definition 2.2. Let $\mathsf{Bimod}\mathcal{A} - \mathcal{A}$ denote the category of $\mathcal{A} - \mathcal{A}$-bimodules. Specifically:

- an object of $\mathsf{Bimod}\mathcal{A}-\mathcal{A}$ is a triple
$$(\mathcal{C}=\{C_{ij}\}_{i,j\in\mathbb{Z}},\{\mu_{ijk}\}_{i,j,k\in\mathbb{Z}},\{\psi_{ijk}\}_{i,j,k\in\mathbb{Z}})$$
where $\mathcal{C}_{ij}$ is an $\mathcal{O}_X$-bimodule and $\mu_{ijk}:\mathcal{C}_{ij}\otimes\mathcal{A}_{jk}\to\mathcal{C}_{ik}$ and $\psi_{ijk}:\mathcal{A}_{ij}\otimes\mathcal{C}_{jk}\to\mathcal{C}_{ik}$ are morphisms of $\mathcal{O}_{X^2}$-modules making $\mathcal{C}$ an $\mathcal{A}$-$\mathcal{A}$ bimodule.
- A morphism $\phi:\mathcal{C}\to\mathcal{D}$ between objects in $\mathsf{Bimod}\mathcal{A}-\mathcal{A}$ is a collection $\phi=\{\phi_{ij}\}_{i,j\in\mathbb{Z}}$ such that $\phi_{ij}:\mathcal{C}_{ij}\to\mathcal{D}_{ij}$ is a morphism of $\mathcal{O}_{X^2}$-modules, and such that ϕ respects the $\mathcal{A}-\mathcal{A}$-bimodule structure on $\mathcal{C}$ and $\mathcal{D}$.

Let $\mathbb{B}$ denote the full subcategory of $\mathsf{Bimod}\mathcal{A}-\mathcal{A}$ whose objects $\mathcal{C}=\{C_{ij}\}_{i,j\in\mathbb{Z}}$ have the property that $\mathcal{C}_{ij}$ is coherent and locally free for all $i,j\in\mathbb{Z}$.

Let $\mathbb{G}\mathrm{r}\mathcal{A}$ denote the full subcategory of $\mathbb{B}$ consisting of objects $\mathcal{C}$ such that for some $n\in\mathbb{Z}$, $\mathcal{C}_{ij}=0$ for $i\neq n$ (we say $\mathcal{C}$ is left-concentrated in degree n).

Definition 2.3. [5, Definition 3.7] Let $\mathcal{C}$ be an object in $\mathbb{B}$ and let $\mathcal{M}$ be a graded right $\mathcal{A}$-module. We define $\underline{\boldsymbol{\mathcal{H}om}}_{\mathsf{Gr}\mathcal{A}}\boldsymbol{(\mathcal{C},\mathcal{M})}$ to be the $\mathbb{Z}$-graded $\mathcal{O}_X$-module whose kth component is the equalizer of the diagram

$$\begin{array}{ccc} \prod_i \mathcal{M}_i\otimes\mathcal{C}^*_{ki} & \xrightarrow{\alpha} & \prod_j \mathcal{M}_j\otimes\mathcal{C}^*_{kj} \\ \beta\downarrow & & \downarrow\gamma \\ \prod_j(\prod_i(\mathcal{M}_j\otimes\mathcal{A}^*_{ij})\otimes\mathcal{C}^*_{ki}) & \xrightarrow[\delta]{} & \prod_j(\prod_i\mathcal{M}_j\otimes(\mathcal{C}_{ki}\otimes\mathcal{A}_{ij})^*) \end{array} \tag{1}$$

where α is the identity map, β is induced by the composition
$$\mathcal{M}_i\xrightarrow{\eta}\mathcal{M}_i\otimes\mathcal{A}_{ij}\otimes\mathcal{A}^*_{ij}\xrightarrow{\mu}\mathcal{M}_j\otimes\mathcal{A}^*_{ij},$$
γ is induced by the dual of
$$\mathcal{C}_{ki}\otimes\mathcal{A}_{ij}\xrightarrow{\mu}\mathcal{C}_{kj},$$
and δ is induced by the composition
$$(\mathcal{M}_j\otimes\mathcal{A}^*_{ij})\otimes\mathcal{C}^*_{ij}\to\mathcal{M}_j\otimes(\mathcal{A}^*_{ij}\otimes\mathcal{C}^*_{ki})\to\mathcal{M}_j\otimes(\mathcal{C}_{ki}\otimes\mathcal{A}_{ij})^*$$
whose left arrow is the associativity isomorphism and whose right arrow is induced by the canonical map [5, Section 2.1]. If $\mathcal{C}$ is an object of $\mathbb{G}\mathrm{r}\mathcal{A}$ left-concentrated in degree k, we define $\boldsymbol{\mathcal{H}om_{\mathsf{Gr}\mathcal{A}}(\mathcal{C},\mathcal{M})}$ to be the equalizer of (1).

Let $\tau:\mathsf{Gr}\mathcal{A}\to\mathsf{Tors}\mathcal{A}$ denote the torsion functor, let $\pi:\mathsf{Gr}\mathcal{A}\to\mathsf{Proj}\mathcal{A}$ denote the quotient functor, and let $\omega:\mathsf{Proj}\mathcal{A}\to\mathsf{Gr}\mathcal{A}$ denote the right adjoint to π. For any $k\in\mathbb{Z}$, let $e_k\mathcal{A}$ denote the right-$\mathcal{A}$-module $\bigoplus_{l\geq k}\mathcal{A}_{kl}$. We define $e_k\mathcal{A}_{\geq k+n}$ to be the sum $\bigoplus_{i\geq 0}e_k\mathcal{A}_{k+n+i}$ and we let $\mathcal{A}_{\geq n}=\bigoplus_k e_k\mathcal{A}_{\geq k+n}$.

Theorem 2.4. *If $\mathcal{M}$ is an object in $\mathsf{Gr}\mathcal{A}$ and $\mathcal{C}$ is an object in $\mathbb{B}$, $\underline{\mathcal{H}om}_{\mathsf{Gr}\mathcal{A}}(\mathcal{C},\mathcal{M})$ inherits a graded right $\mathcal{A}$-module structure from the left $\mathcal{A}$-module structure of $\mathcal{C}$, making $\underline{\mathcal{H}om}_{\mathsf{Gr}\mathcal{A}}(-,-):\mathbb{B}^{op}\times\mathsf{Gr}\mathcal{A}\to\mathsf{Gr}\mathcal{A}$ a bifunctor.*

Furthermore

(1) $\tau(-)\cong\lim_{n\to\infty}\underline{\mathcal{H}om}_{\mathsf{Gr}\mathcal{A}}(\mathcal{A}/\mathcal{A}_{\geq n},-)$,

(2) *If $\mathcal{F}$ is a coherent, locally free $\mathcal{O}_X$-bimodule,*
$$\mathcal{H}om_{\mathsf{Gr}\mathcal{A}}(\mathcal{F}\otimes e_k\mathcal{A},-)\cong(-)_k\otimes\mathcal{F}^*$$
and

(3) *If $\mathcal{L}$ is an $\mathcal{O}_X$-module and $\mathcal{M}$ is an object of* $\mathsf{Gr}\mathcal{A}$,

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{H}om_{\mathsf{Gr}\mathcal{A}}(e_k\mathcal{A}, \mathcal{M})) \cong \mathrm{Hom}_{\mathsf{Gr}\mathcal{A}}(\mathcal{L} \otimes e_k\mathcal{A}, \mathcal{M}).$$

Proof. The first statement is [5, Proposition 3.11], (1) is [5, Proposition 3.19], (2) is [5, Theorem 3.16(4)] and (3) is a consequence of [5, Proposition 3.10] □

By Theorem 2.4 (2), $\mathcal{H}om_{\mathsf{Gr}\mathcal{A}}(-, \mathcal{M})$ is $\mathcal{F} \otimes e_k\mathcal{A}$-acyclic when $\mathcal{F}$ is a coherent, locally free $\mathcal{O}_X$-bimodule. Thus, one may use the resolution [8, Theorem 7.1.2] to compute the derived functors of $\mathcal{H}om_{\mathsf{Gr}\mathcal{A}}(\mathcal{A}/\mathcal{A}_{\geq 1}, -)$. By Theorem 2.4(1), we may thus compute the derived functors of τ:

Theorem 2.5. *The cohomological dimension of τ is 2. For $i < 2$ and $\mathcal{L}$ a coherent, locally free $\mathcal{O}_X$-module,*

$$\mathrm{R}^i\,\tau(\mathcal{L} \otimes e_k\mathcal{A}) = 0$$

and

$$(\mathrm{R}^2\,\tau(\mathcal{L} \otimes e_l\mathcal{A}))_{l-2-i} \cong \begin{cases} \mathcal{L} \otimes \mathcal{A}^*_{l-2-i,l-2} & \textit{if } i \geq 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The first result is [5, Corollary 4.10], while the remainder is [5, Lemma 4.9]. □

3. Serre finiteness and Serre vanishing

In this section let I denote a finite subset of $\mathbb{Z} \times \mathbb{Z}$. The proof of the following lemma is straightforward, so we omit it.

Lemma 3.1. *If $\mathcal{M}$ is a noetherian object in* $\mathsf{Gr}\mathcal{A}$*, $\pi\mathcal{M}$ is a noetherian object in* $\mathsf{Proj}\mathcal{A}$ *and $\mathcal{M}$ is locally coherent.*

Lemma 3.2. *If $\mathcal{M}$ is a noetherian object in* $\mathsf{Gr}\mathcal{A}$*, $\mathrm{R}^i\tau\mathcal{M}$ is locally coherent for all $i \geq 0$.*

Proof. The module $\mathcal{O}_X(j) \otimes e_k\mathcal{A}$ is noetherian by [5, Lemma 2.17] and the lemma holds with $\mathcal{M} = \bigoplus_{(j,k)\in I} \mathcal{O}_X(j) \otimes e_k\mathcal{A}$ by Theorem 2.5.

To prove the result for arbitrary noetherian $\mathcal{M}$, we use descending induction on i. For $i > 2$, $\mathrm{R}^i\tau\mathcal{M} = 0$ by Theorem 2.5, so the result is trivial in this case. Since $\mathcal{M}$ is noetherian, there is a finite subset $I \subset \mathbb{Z} \times \mathbb{Z}$ and a short exact sequence

$$0 \to \mathcal{R} \to \bigoplus_{(j,k)\in I} \mathcal{O}_X(j) \otimes e_k\mathcal{A} \to \mathcal{M} \to 0$$

by [5, Lemma 2.17]. This induces an exact sequence of $\mathcal{A}$-modules

$$\ldots \to (\mathrm{R}^i\tau(\bigoplus_{(j,k)\in I} \mathcal{O}_X(j) \otimes e_k\mathcal{A}))_l \to (\mathrm{R}^i\tau\mathcal{M})_l \to (\mathrm{R}^{i+1}\tau\mathcal{R})_l \to \ldots$$

The left module is coherent by the first part of the proof, while the right module is coherent by the induction hypothesis. Hence the middle module is coherent since X is noetherian. □

Corollary 3.3. *If $\mathcal{M}$ is a noetherian object in* $\mathsf{Gr}\mathcal{A}$*, $\mathrm{R}^i(\omega(-)_k)(\pi\mathcal{M})$ is coherent for all $i \geq 0$ and all $k \in \mathbb{Z}$.*

Proof. Since $(-)_k : \mathsf{Gr}\mathcal{A} \to \mathsf{Mod}X$ is an exact functor, $\mathrm{R}^i(\omega(-)_k)(\pi\mathcal{M}) \cong \mathrm{R}^i\omega(\pi\mathcal{M})_k$.

Now, to prove $\omega(\pi\mathcal{M})_k$ is coherent, we note that there is an exact sequence in $\mathsf{Mod}X$

$$0 \to \tau\mathcal{M}_k \to \mathcal{M}_k \to \omega(\pi\mathcal{M})_k \to (\mathrm{R}^1\tau\mathcal{M})_k \to 0$$

by [5, Theorem 4.11]. Since $\mathcal{M}_k$ and $(\mathrm{R}^1\tau\mathcal{M})_k$ are coherent by Lemma 3.1 and Lemma 3.2 respectively, $\omega(\pi\mathcal{M})_k$ is coherent since X is noetherian.

The fact that $\mathrm{R}^i\,\omega(\pi\mathcal{M})_k$ is coherent for $i > 0$ follows from Lemma 3.2 since, in this case,

$$(\mathrm{R}^i\,\omega(\pi\mathcal{M}))_k \cong (\mathrm{R}^{i+1}\,\tau\mathcal{M})_k \tag{2}$$

by [5, Theorem 4.11]. □

Lemma 3.4. *For $\mathcal{N}$ noetherian in $\mathsf{Gr}\mathcal{A}$, $\mathrm{R}^1\,\omega(\pi\mathcal{N})_k = 0$ for $k >> 0$.*

Proof. When $\mathcal{N} = \bigoplus_{(l,m)\in I} (\mathcal{O}_X(l) \otimes e_m\mathcal{A})$, the result follows from (2) and Theorem 2.5.

More generally, there is a short exact sequence

$$0 \to \mathcal{R} \to \pi(\bigoplus_{(l,m)\in I} \mathcal{O}_X(l) \otimes e_m\mathcal{A}) \to \pi\mathcal{N} \to 0$$

which induces an exact sequence

$$\cdots \to \mathrm{R}^1\,\omega(\pi(\bigoplus_{(l,m)\in I} \mathcal{O}_X(l) \otimes e_m\mathcal{A})) \to \mathrm{R}^1\,\omega(\pi\mathcal{N}) \to \mathrm{R}^2\,\omega(\mathcal{R}) = 0.$$

where the right equality is due to (2) and Theorem 2.5. Since the left module is 0 in high degree, so is $\mathrm{R}^1\,\omega(\pi\mathcal{N})$. □

Theorem 3.5. *For any noetherian object $\mathcal{N}$ in $\mathsf{Gr}\mathcal{A}$,*

(1) $\mathrm{Ext}^i_{\mathsf{Proj}\mathcal{A}}(\bigoplus_{(j,k)\in I} \pi(\mathcal{O}_X(j) \otimes e_k\mathcal{A}), \pi\mathcal{N})$ *is finite-dimensional over K for all $i \geq 0$, and*

(2) *for $i > 0$,* $\mathrm{Ext}^i_{\mathsf{Proj}\mathcal{A}}(\bigoplus_{(j,k)\in I} \pi(\mathcal{O}_X(j) \otimes e_k\mathcal{A}), \pi\mathcal{N}) = 0$ *whenever $j << 0$ and $k >> 0$.*

Proof. Let d denote the cohomological dimension of X. Since $\mathrm{Ext}^i_{\mathsf{Proj}\mathcal{A}}(-, \pi\mathcal{N})$ commutes with finite direct sums, it suffices to prove the theorem when I has only one element.

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Proj}\mathcal{A}}(\pi(\mathcal{O}_X(j) \otimes e_k\mathcal{A}), \pi\mathcal{N}) &\cong \mathrm{Hom}_{\mathsf{Gr}\mathcal{A}}(\mathcal{O}_X(j) \otimes e_k\mathcal{A}, \omega\pi\mathcal{N}) \\
&\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(j), \mathcal{H}om_{\mathsf{Gr}\mathcal{A}}(e_k\mathcal{A}, \omega\pi\mathcal{N})) \\
&\cong \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(j), \omega(\pi\mathcal{N})_k) \\
&\cong \Gamma(\mathcal{O}_X(-j) \otimes \omega(-)_k)(\pi\mathcal{N})
\end{aligned}$$

where the second isomorphism is from Theorem 2.4 (3), while the third isomorphism is from Theorem 2.4 (2). Thus,

$$\mathrm{Ext}^i_{\mathsf{Proj}\mathcal{A}}(\pi(\mathcal{O}_X(j) \otimes e_k\mathcal{A}), \pi\mathcal{N}) \cong \mathrm{R}^i(\Gamma \circ (\mathcal{O}_X(-j) \otimes \omega(-)_k))\pi\mathcal{N}.$$

If $i = 0$, (1) follows from Corollary 3.3 and [1, III, Theorem 5.2a, p. 228].

If $0 < i < d+1$, the Grothendieck spectral sequence gives us an exact sequence

$$\ldots \to \mathrm{R}^i\,\Gamma(\mathcal{O}_X(-j) \otimes \omega(\pi\mathcal{N})_k) \to \mathrm{R}^i(\Gamma \circ \mathcal{O}_X(-j) \otimes \omega(-)_k)\pi\mathcal{N} \to \tag{3}$$

$$\mathrm{R}^{i-1}\,\Gamma\,\mathrm{R}^1(\mathcal{O}_X(-j)\otimes\omega(-)_k)\pi\mathcal{N}\to\ldots$$

Since $\omega(\pi\mathcal{N})_k$ and $\mathrm{R}^1(\mathcal{O}_X(-j)\otimes\omega(-)_k)\pi\mathcal{N}\cong\mathcal{O}_X(-j)\otimes\mathrm{R}^1(\omega(-)_k)\pi\mathcal{N}$ are coherent by Corollary 3.3, the first and last terms of (3) are finite-dimensional by [1, III, Theorem 5.2a, p.228]. Thus, the middle term of (3) is finite-dimensional as well, which proves (1) in this case. To prove (2) in this case, we note that, since $\omega(\pi\mathcal{N})_k$ is coherent, the first module of (3) is 0 for $j << 0$ by [1, III, Theorem 5.2b, p.228]. If $i > 1$, the last module of (3) is 0 for $j << 0$ for the same reason. Finally, if $i = 1$, the last module of (3) is 0 since $\mathrm{R}^1\,\omega(\pi\mathcal{N})_k = 0$ for $k >> 0$ by Lemma 3.4.

If $i = d + 1$, the Grothendieck spectral sequence gives an isomorphism

$$\mathrm{R}^{d+1}(\Gamma\circ(\mathcal{O}_X(-j)\otimes\omega(-)_k)\pi\mathcal{N}\cong\mathrm{R}^d\,\Gamma\,\mathrm{R}^1(\mathcal{O}_X(-j)\otimes\omega(-)_k)\pi\mathcal{N}.$$

In this case, (1) again follows from Corollary 3.3 and [1, III, Theorem 5.2a, p.228], while (2) follows from Lemma 3.4. □

Corollary 3.6. *If $\mathcal{M}$ and $\mathcal{N}$ are noetherian objects in $\mathsf{Gr}\mathcal{A}$, $\mathrm{Ext}^i_{\mathsf{Proj}\mathcal{A}}(\pi\mathcal{M},\pi\mathcal{N})$ is finite-dimensional for $i \geq 0$.*

Proof. Since $\mathcal{M}$ is noetherian, there is an exact sequence

$$0\to\mathcal{R}\to\pi(\bigoplus_{(j,k)\in I}\mathcal{O}_X(j)\otimes e_k\mathcal{A})\to\pi\mathcal{M}\to 0.$$

Since the central term is noetherian by Lemma 3.1, so is the $\mathcal{R}$. Since $\mathrm{Hom}_{\mathsf{Proj}\mathcal{A}}(-,\pi\mathcal{N})$ is left exact, there are exact sequences

$$(4)\qquad 0\to\mathrm{Hom}_{\mathsf{Proj}\mathcal{A}}(\pi\mathcal{M},\pi\mathcal{N})\to\mathrm{Hom}_{\mathsf{Proj}\mathcal{A}}(\pi(\bigoplus_{(j,k)\in I}\mathcal{O}_X(j)\otimes e_k\mathcal{A}),\pi\mathcal{N})\to$$

and, for $i \geq 1$,

(5)

$$\to\mathrm{Ext}^{i-1}_{\mathsf{Proj}\mathcal{A}}(\mathcal{R},\pi\mathcal{N})\to\mathrm{Ext}^i_{\mathsf{Proj}\mathcal{A}}(\pi\mathcal{M},\pi\mathcal{N})\to\mathrm{Ext}^i_{\mathsf{Proj}\mathcal{A}}(\pi(\bigoplus_{(j,k)\in I}\mathcal{O}_X(j)\otimes e_k\mathcal{A}),\pi\mathcal{N})\to$$

Since π commutes with direct sums, the right-hand terms of (4) and (5) are finite-dimensional by Theorem 3.5(1), while the left hand term of (5) is finite-dimensional by the induction hypothesis. □

4. Riemann-Roch and Adjunction

Let X be a smooth projective curve, let $\mathcal{A}$ be the noncommutative symmetric algebra generated by a locally free $\mathcal{O}_X$-bimodule $\mathcal{E}$ of rank 2, and let $Y = \mathsf{Proj}\mathcal{A}$. In this section, we state the Riemann-Roch theorem and adjunction formula for Y. In order to state these results, we need to define an intersection multiplicity on Y. This definition depends on the fact that Y has well behaved cohomology, so we begin this section by reviewing relevant facts regarding the cohomology of Y.

Let $\mathcal{O}_Y = \pi\,\mathrm{pr}_{2*}\,e_0\mathcal{A}$. By [5, Theorem 5.20], Y satisfies Serre duality, i.e., there exists an object ω_Y in $\mathsf{Proj}\mathcal{A}$, called the canonical sheaf on Y, such that

$$(6)\qquad \mathrm{Ext}^{2-i}_Y(\mathcal{O}_Y,-)'\cong\mathrm{Ext}^i_Y(-,\omega_Y)$$

for all $0 \leq i \leq 2$.

By [5, Theorem 4.16], Y has cohomological dimension two, i.e.

$$(7)\qquad 2=\sup\{i|\,\mathrm{Ext}^i_Y(\mathcal{O}_Y,\mathcal{M})\neq 0\text{ for some noetherian object }\mathcal{M}\text{ in }\mathsf{Proj}\mathcal{A}\}.$$

We write $D : Y \to Y$ for an autoequivalence, $-D : Y \to Y$ for the inverse of D, and $\mathcal{M}(D) := D(\mathcal{M})$ for $\mathcal{M} \in Y$.

Definition 4.1. [2, Definition 2.3] A **weak divisor** on Y is an element $\mathcal{O}_D \in K_0(Y)$ of the form $\mathcal{O}_D = [\mathcal{O}_Y] - [\mathcal{O}_Y(-D)]$ for some autoequivalence D of Y.

We now define an intersection multiplicity on Y following [2]. Let $\mathcal{M}$ be a noetherian object in $\mathsf{Proj}\mathcal{A}$, and let $[\mathcal{M}]$ denote its class in $K_0(Y)$. We define a map $\xi(-,\omega_Y) : K_0(Y) \to \mathbb{Z}$ by

$$\xi([\mathcal{M}],\omega_Y) = \sum_{i=0}^{\infty}(-1)^i \dim_K \operatorname{Ext}^i_Y(\mathcal{M},\omega_Y).$$

This map is well defined by (6), (7) and Corollary 3.6. If $\mathcal{O}_D$ is a weak divisor on Y, we define a map $\xi(\mathcal{O}_D,-) : K_0(Y) \to \mathbb{Z}$ by

$$\xi(\mathcal{O}_D,[\mathcal{M}]) = \sum_{i=0}^{\infty}(-1)^i(\dim_K \operatorname{Ext}^i_Y(\mathcal{O}_Y,\mathcal{M}) - \dim_K \operatorname{Ext}^i_Y(\mathcal{O}_Y(-D),\mathcal{M})).$$

This map is well defined by (7) and Corollary 3.6. We define the intersection multiplicity of $\mathcal{M}$ and ω_Y by

$$\mathcal{M}\cdot\omega_Y := (-1)^{\operatorname{codim}\mathcal{M}}\xi([\mathcal{M}],\omega_Y)$$

for some suitably defined integer $\operatorname{codim}\mathcal{M}$. Thus, if $\mathcal{M}$ is a "curve" on Y, then $\mathcal{M}\cdot\omega_Y = -\xi([\mathcal{M}],\omega_Y)$. Similarly, we define the intersection multiplicity of $\mathcal{O}_D$ and $\mathcal{M}$ by

$$\mathcal{O}_D\cdot\mathcal{M} := -\xi(\mathcal{O}_D,[\mathcal{M}]).$$

We have yet to establish that ω_Y is noetherian. However, if it is noetherian, notice that $\mathcal{O}_D\cdot\omega_Y$ is defined unambiguously.

Finally, we define a map $\chi(-) : K_0(Y) \to \mathbb{Z}$ by

$$\chi([\mathcal{M}]) := \sum_{i=0}^{\infty}(-1)^i \dim_K \operatorname{Ext}^i_Y(\mathcal{O}_Y,\mathcal{M}).$$

Corollary 4.2. *Let $Y = \mathsf{Proj}\mathcal{A}$, let ω_Y denote the canonical sheaf on Y, and suppose $\mathcal{O}_D$ is a weak divisor on Y. Then we have the following formulas:*

(1) *(Riemann-Roch)*

$$\chi(\mathcal{O}_Y(D)) = \frac{1}{2}(\mathcal{O}_D\cdot\mathcal{O}_D - \mathcal{O}_D\cdot\omega_Y + \mathcal{O}_D\cdot\mathcal{O}_Y) + 1 + p_a$$

where $p_a := \chi([\mathcal{O}_Y]) - 1$ is the arithmetic genus of Y.

(2) *(Adjunction)*

$$2g-2 = \mathcal{O}_D\cdot\mathcal{O}_D + \mathcal{O}_D\cdot\omega_Y - \mathcal{O}_D\cdot\mathcal{O}_Y$$

where $g := 1-\chi(\mathcal{O}_D)$ is the genus of $\mathcal{O}_D$.

Proof. The quasi-scheme Y is Ext-finite by Corollary 3.6, has cohomological dimension 2 by [5, Theorem 4.16], and satisfies Serre duality with ω_Y by [5, Theorem 5.20]. Thus, Y is classical Cohen-Macaulay, and the result follows [2, Theorem 3.11]. □

In stating the Corollary, we defined the intersection multiplicity only for specific elements of $K_0(Y)\times K_0(Y)$. In order to define an intersection multiplicity on the entire set $K_0(Y)\times K_0(Y)$, one must first prove that Y has finite homological dimension. In [3, Section 6], Mori and Smith study noncommutative $\mathbb{P}^1$-bundles $Y = \mathsf{Proj}\mathcal{A}$ such that $\mathcal{A}$ is generated by a bimodule $\mathcal{E}$ with the property that $\mathcal{E}\otimes\mathcal{E}$

contains a nondegenerate invertible bimodule. In this case, they use the structure of $K_0(Y)$ to prove that Y has finite homological dimension. They then compute various intersections on Y without the use of either the Riemann-Roch theorem or the adjunction formula. In particular, they prove that distinct fibers on Y do not meet, and that a fiber and a section on Y meet exactly once.

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