Maps to Noncommutative Projective Spaces (w/ Daniel Chan)

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Conventions

- always work over a field k
- unless otherwise stated, work with right modules
- always let C denote a k-linear Hom-finite abelian category.

Part 1

Maps to Projective Spaces

Maps from line bundles

Suppose

- X is a projective variety
- $\mathcal{L}$ is line-bundle on X gen. by $n+1$ global sections.

$$\text{Given } (X, \mathcal{L}),\ \exists \text{ morphism } f : X \to \mathbb{P}^n.$$

Stein factorization of f

f factors as

$$X \xrightarrow{g} \text{Proj } \Gamma_*(X, \mathcal{L}) \xrightarrow{h} \text{Proj } \mathbb{S}(\Gamma(X, \mathcal{L})) = \mathbb{P}^n$$

where g proper, h finite.

Goal

Generalize above construction to produce maps from nc elliptic curves to nc projective spaces.

Examples

Artin-Zhang (1994) and Polishchuk (2005) study nc generalizations of g.

Elliptic curves in noncommutative projective planes

X a smooth elliptic curve. Artin, Tate and Van den Bergh construct closed immersions $f : X \to \mathbb{P}^2_{nc}$.

Theorem (S.P. Smith (2003))

If A is a loc. finite noetherian $\mathbb{N}$-graded algebra and J is a graded ideal, then $A \to A/J$ induces closed immersion of noncommutative spaces

$$\text{Proj}A/J \to \text{Proj}A.$$

Double covers of $\mathbb{P}^1$

X a smooth elliptic curve. $\mathcal{L} =$ deg. 2 line bundle over X. $\mathcal{L}$ induces double cover $X \to \mathbb{P}^1$. No (very) nc analogue.

Data associated to $(X, \mathcal{L})$

Given $(X, \mathcal{L})$, we can construct:

- a canonical finite map $\mathbb{S}(\Gamma(X, \mathcal{L})) \to \Gamma_*(X, \mathcal{L})$,
- an induced finite morphism
  $\operatorname{Proj} \Gamma_*(X, \mathcal{L}) \xrightarrow{h} \operatorname{Proj} \mathbb{S}(\Gamma(X, \mathcal{L}))$, and
- pullback of Koszul complex over $\mathbb{P}^n$ to X

$$0 \to \bigwedge^{n+1} V \otimes \mathcal{L}^{-n-1} \to \cdots \to \bigwedge^{1} V \otimes \mathcal{L}^{-1} \to \mathcal{O}_X \to 0$$

is **exact**, where $V = \Gamma(X, \mathcal{L})$.

Koszul Complex

Let $V = \Gamma(\mathbb{P}^n, \mathcal{O}(1))$. $\exists$ exact sequence

$$0 \to \bigwedge^{n+1} V \otimes \mathcal{O}_{\mathbb{P}^n}(-n-1) \to \cdots \to \bigwedge^{1} V \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to 0$$

An Example

D, E Noncommutative Spaces

D $\xrightarrow{f}$ E denotes adjoint pair (f^*, f_*) in the diagram

$$\mathsf{D} \underset{f^*}{\overset{f_*}{\rightleftharpoons}} \mathsf{E}$$

Motivation

If $f : Y \to X$ is a morphism of commutative schemes, (f^*, f_*) is an adjoint pair.

Notion is too general.

Define $\text{Qcoh}\mathbb{P}^0 \underset{f^*}{\overset{f_*}{\rightleftharpoons}} \text{Qcoh}\mathbb{P}^1$ by $f^* = \mathrm{H}^1(\mathbb{P}^1, -)$. Then f^* is not exact on ses of vector-bundles so can't come from a map of schemes.

Part 2

Maps to Noncommutative Projective Spaces

Replacement for $\mathcal{L}$ (Polishchuk (2005))

Let X be a variety, $\mathcal{L}$ a line bundle on X. Recall

$$\Gamma_*(X, \mathcal{L}) = \bigoplus_{n \geq 0} \Gamma(X, \mathcal{L}^{\otimes n})$$

depends on monoidal structure on $\mathrm{Coh} X$.

Categories natural in nc algebraic geometry (e.g. $\mathrm{Mod} R$) may not have a monoidal structure.

Artin-Zhang (1994)

Given $\mathcal{A} \in \mathrm{ob}\, \mathsf{C}$, consider $s^i(\mathcal{A})$ where s is autoequivalence of C.

Bondal-Polishchuk (1993), Polishchuk (2005)

Let $\underline{\mathcal{L}} := (\mathcal{L}_i)_{i \in \mathbb{Z}}$ where $\mathcal{L}_i \in \mathrm{Ob}\, \mathsf{C}$

Question

How do you form a ring from a sequence $(\mathcal{L}_i)_{i \in \mathbb{Z}}$ of objects of C?

$\mathbb{Z}$-algebras (Bondal and Polishchuk (1993))

A $\mathbb{Z}$-algebra is ring A with vector space decomposition $\bigoplus_{i,j\in\mathbb{Z}} A_{ij}$ such that

- $A_{ij}A_{jk} \subset A_{ik}$,
- $A_{ij}A_{kl} = 0$ for $k \neq j$, and
- A_{ii} contains a unit e_i so that $e_i A = \bigoplus_j A_{ij}$.

Periodicity (Sierra (2011))

Periodic $\mathbb{Z}$-algebras generalize $\mathbb{Z}$-graded algebras

Let A be a $\mathbb{Z}$-algebra. Let $A(\ell)$ be the $\mathbb{Z}$-algebra with

$$A(\ell)_{ij} := A_{i+\ell,j+\ell}$$

A is *ℓ-periodic* if $A \cong A(\ell)$ as algebras.

Observation (Sierra (2011))

If A is a 1-periodic $\mathbb{Z}$-algebra, then A is Morita equivalent to a $\mathbb{Z}$-graded algebra.

Replacement for $\Gamma_*(X, \mathcal{L})$ (Polishchuk (2005))

Let $\underline{\mathcal{L}} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$. Let $(B_{\underline{\mathcal{L}}})_{ij} := \mathrm{Hom}_C(\mathcal{L}_{-j}, \mathcal{L}_{-i})$. Then $B_{\underline{\mathcal{L}}}$ with mult. induced by composition, is a $\mathbb{Z}$-algebra.

The $\mathbb{Z}$-algebra $B_{\underline{\mathcal{L}}}$ plays the role of $\Gamma_*(X, \mathcal{L})$

Motivation

Let $\mathcal{L}_i := \mathcal{L}^{\otimes i}$. Then $B_{\underline{\mathcal{L}}}$ is 1-periodic and

$$\mathrm{Gr} B_{\underline{\mathcal{L}}} \equiv \mathrm{Gr} \Gamma_*(X, \mathcal{L}).$$

Replacement for $\mathbb{S}(\Gamma(X, \mathcal{L}))$

The noncommutative symmetric algebra of $\underline{\mathcal{L}}$

We define $A_{\underline{\mathcal{L}}}$ to be quadratic part of $B_{\underline{\mathcal{L}}}$.

By construction, there is a morphism of $\mathbb{Z}$-algebras

$$A_{\underline{\mathcal{L}}} \to B_{\underline{\mathcal{L}}}.$$

analogous to

$$\mathbb{S}(\Gamma(X, \mathcal{L})) \longrightarrow \Gamma_*(X, \mathcal{L})$$

Relationship to Van den Bergh's $\mathbb{S}^{nc}(V)$

Necessary and sufficient conditions on $\underline{\mathcal{L}}$ are known (N (2019)) to ensure

$$A_{\underline{\mathcal{L}}} \cong \mathbb{S}^{nc}(V).$$

Quadratic Duals (Bondal-Polishchuk (1993))

Definition of $A^!$

- $A =$ locally finite, quadratic $\mathbb{Z}$-algebra with relns I.
- Define $A^! =$ quadratic $\mathbb{Z}$-algebra with gens

$$A^!_{i+1,i} := A^*_{i,i+1}$$

with relations the kernel of

$$A^!_{i+2,i+1} \otimes A^!_{i+1,i} \cong (A_{i,i+1} \otimes A_{i+1,i+2})^* \to I^*_{i,i+2}$$

induced by inclusion $I_{i,i+2} \to A_{i,i+1} \otimes A_{i+1,i+2}$.

Motivating Example

In $\mathbb{Z}$-graded case, we have $\mathbb{S}(V)^! = \bigwedge(V^*)$

The Koszul complex of $\underline{\mathcal{L}}$

$\underline{\mathcal{L}}$ = sequence of objects in C with $\text{End}(\mathcal{L}_i) = k$ for all i, $A := A_{\underline{\mathcal{L}}}$.
There is a complex of form

$$\cdots \to A^{!*}_{j+2,j} \otimes \mathcal{L}_{-j-2} \to A^{!*}_{j+1,j} \otimes \mathcal{L}_{-j-1} \to A^{!*}_{j,j} \otimes \mathcal{L}_{-j} \to 0$$

Evaluation is map $\text{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \cong \bigoplus \mathcal{E} \overset{(f_1,\ldots,f_n)}{\longrightarrow} \mathcal{F}$.

Sample map $A^{!*}_{2,0} \otimes \mathcal{L}_{-2} \longrightarrow A^{!*}_{1,0} \otimes \mathcal{L}_{-1}$

$$\begin{aligned} A^{!*}_{2,0} \otimes \mathcal{L}_{-2} &\longrightarrow A_{0,1} \otimes A_{1,2} \otimes \mathcal{L}_{-2} \\ &\overset{=}{\longrightarrow} A_{0,1} \otimes \text{Hom}(\mathcal{L}_{-2}, \mathcal{L}_{-1}) \otimes \mathcal{L}_{-2} \\ &\overset{eval}{\longrightarrow} A_{0,1} \otimes \mathcal{L}_{-1} \\ &\overset{\cong}{\longrightarrow} A^{!*}_{1,0} \otimes \mathcal{L}_{-1} \end{aligned}$$

Helices

Definition of Helix (Chan-N (2022))

A sequence $\underline{\mathcal{L}} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$ of objects in C is a **helix of length** n if for all i, j,

- there exists an $m \geq 0$ such that for all $l \geq m$, $\text{Ext}^j(\mathcal{L}_i, \mathcal{L}_{i+l}) = 0$ for all $j > 0$ (Serre vanishing).
- $\text{End}(\mathcal{L}_i) = k$ (i.e. $\mathcal{L}_i$ is "simple"), and
- there are f.d. vector spaces $V_{j+3,j}, \ldots, V_{j+n,j}$ and exact sequences whose right three terms are the Koszul complex

$$0 \longrightarrow V_{j+n,j} \otimes \mathcal{L}_{-j-n} \longrightarrow \cdots \longrightarrow V_{j+3,j} \otimes \mathcal{L}_{-j-3} \longrightarrow$$

$$A^{!*}_{j+2,j} \otimes \mathcal{L}_{-j-2} \xrightarrow{\phi_2} A^{!*}_{j+1,j} \otimes \mathcal{L}_{-j-1} \xrightarrow{\phi_1} A^{!*}_{j,j} \otimes \mathcal{L}_{-j} \longrightarrow 0$$

where $A = A_{\underline{\mathcal{L}}}$.

The map of noncommutative spaces induced by a helix

Let $(B_{\underline{\mathcal{L}}})_{\geq 0} =: B$.

Theorem (Chan-N (2022))

If $\underline{\mathcal{L}}$ is a helix of length n, then

1. the canonical map
$$\psi : A_{\underline{\mathcal{L}}} \to B$$
makes Be_j a finitely generated $A_{\underline{\mathcal{L}}}$-module for all j, and
2. the map ψ descends to an adjoint pair
$$\mathrm{Proj} B \leftrightarrows \mathrm{Proj} A_{\underline{\mathcal{L}}}.$$

Recall: $\mathrm{Tors} B$ =full subcategory of objects in $\mathrm{Gr} B$ whose elements generate right-bounded modules.

$$\mathrm{Proj} B := \mathrm{Gr} B / \mathrm{Tors} B.$$

Part 3

Interlude: Noncommutative Elliptic Curves

Conventions for remainder of talk

- $k = \mathbb{C}$
- X is smooth elliptic curve (over k)
- $\operatorname{Coh} X$ is category of coherent sheaves over X

Classification of vector bundles over X (Atiyah (1957))

Let $E(r,d)$ =set of iso. classes of indecomposable vector bundles of rank r and degree d.

Theorem (Atiyah (1957))

For each $r \geq 1$ and each $d \in \mathbb{Z}$, $E(r,d)$ is parameterized by the points of X.

- A bundle $\mathcal{E}$ in $\mathrm{Coh}\mathbb{P}^1$ is simple if and only if $\mathcal{E}$ is a line bundle.
- A bundle $\mathcal{E}$ in $E(r,d)$ is simple if and only if $\gcd(r,d) = 1$.

We will construct helices (of length 2 and 3) whose terms are simple vector bundles over X (not nec. line bundles)

Let A be coherent connected $\mathbb{Z}$-algebra and let

- $\mathrm{coh}A =$ cat. of (graded right) coherent modules
- $\mathrm{tors}A =$ full subcat. of right-bounded modules.

Definition (Polishchuk (2005))

$\mathrm{cohproj}A := \mathrm{coh}A/\mathrm{tors}A$

Remark

If A is noetherian, $\mathrm{cohproj}A \equiv \mathrm{proj}A$.

Noncommutative elliptic curves (Polishchuk (2002))

Theorem (Polishchuk (2002))

For each $\theta \in \mathbb{R}$, $\exists$ t-structure on $D^b(X)$ w/heart C^θ such that

- $D^b(C^\theta) \equiv D^b(X)$,
- C^θ has cohomological dimension 1, and
- if θ is irrational, then every nonzero object in C^θ is nonnoetherian.

Theorem (Polishshcuk (2002))

If $\underline{\mathcal{L}} = (\mathcal{L}_i)_{i\in\mathbb{Z}}$ is a sequence of simple bundles such that $\mu(\mathcal{L}_m) > \theta$ for all m and $\lim_{m\to-\infty}\mu(\mathcal{L}_m) = \theta$, then

$$C^\theta \equiv \text{cohproj} B_{\underline{\mathcal{L}}}.$$

Part 4

First Application: Maps to $\mathbb{P}^1_d$

Piontkovski's noncommutative projective line $\mathbb{P}^1_d$

Theorem (Zhang (1998))

If A is connected, gen. in degree 1 and regular of dim 2 then

$$A \cong k\langle x_1, \ldots, x_n\rangle / \langle b\rangle$$

where $n \geq 2$, $b = \sum_{i=1}^{n} x_i \sigma(x_{n-i+1})$ and $\sigma \in \text{Aut } k\langle x_1, \ldots, x_n\rangle$. If $n > 2$, A is non-noetherian.

Theorem (Piontkovski (2008))

$n > 2$ implies A is coherent. If $\mathbb{P}^1_n := \text{cohproj}A$, then $\mathbb{P}^1_n$ depends only on n. Furthermore, $\mathbb{P}^1_2 \equiv \text{Coh}\mathbb{P}^1$.

Example: Maps from Elliptic Curves to Projective Lines

Double cover of $\mathbb{P}^1$

$\mathcal{L} =$ degree 2 line bundle over X.

- $\mathcal{L}$ induces double cover

$$X \cong \text{Proj } \Gamma_*(X, \mathcal{L}) \xrightarrow{h} \text{Proj } \mathbb{S}(\Gamma(X, \mathcal{L})) \cong \mathbb{P}^1$$

  ramified at 4 points.
- Pullback of Koszul complex over $\mathbb{P}^1$ takes form

$$0 \longrightarrow \mathcal{L}^{-2} \longrightarrow \text{Hom}(\mathcal{O}_X, \mathcal{L}) \otimes \mathcal{L}^{-1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Goal

Look for interesting helices over $\mathsf{C} = \text{Coh}X$ with the same "shape" as this example.

Kuleshov's Lemma (Kuleshov (1992))

Definition (Kuleshov (1992))

$(\mathcal{E}, \mathcal{F})$ is *simple pair* if $\mathcal{E}$ and $\mathcal{F}$ are simple and exactly one of $\mathrm{Hom}(\mathcal{E}, \mathcal{F})$, $\mathrm{Ext}^1(\mathcal{E}, \mathcal{F})$ is nonzero.

Lemma (Kuleshov (1992))

Let $\mathcal{E}_1$ be a simple bundle. If

$$0 \longrightarrow \mathcal{L} \longrightarrow V \otimes \mathcal{E}_1 \longrightarrow \mathcal{E}_2 \longrightarrow 0$$

is exact, then TFAE:

1. $(\mathcal{L}, \mathcal{E}_1)$ is a simple pair and $V \cong \mathrm{Hom}(\mathcal{L}, \mathcal{E}_1)^*$.
2. $(\mathcal{E}_1, \mathcal{E}_2)$ is a simple pair and $V \cong \mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$.

Our idea

Use Lemma to construct a helix starting from two simple bundles. Will need $\mathcal{L} \overset{coev}{\to} \mathrm{Hom}(\mathcal{L}, \mathcal{E})^* \otimes \mathcal{E}$ to be injective.

Modification of Kuleshov's Lemma

Injective pairs

A simple pair $(\mathcal{E}, \mathcal{F})$ of bundles is an *injective pair* if $\mathrm{Ext}^1(\mathcal{E}, \mathcal{F}) = 0$ and every nonzero map $\mathcal{E} \to \mathcal{F}$ is injective.

Lemma (Chan-N (2021))

Let $(\mathcal{L}_0, \mathcal{L}_1)$ be an injective pair of bundles such that $d := \dim \mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1) > 1$. Then the ses

$$0 \to \mathcal{L}_0 \overset{coev}{\to} \mathrm{Hom}(\mathcal{L}_0, \mathcal{L}_1)^* \otimes \mathcal{L}_1 \to \mathcal{L}_2 \to 0$$

defines an injective pair of bundles $(\mathcal{L}_1, \mathcal{L}_2)$.

Construction of $\underline{\mathcal{L}}$

Start with $\mathcal{L}_0 \in E(1,0)$, $\mathcal{L}_1 \in E(1,d)$. Lemma gives $(\mathcal{L}_i)_{i \geq 0}$. Do the same starting with the injective pair $(\mathcal{L}_1^*, \mathcal{L}_0^*)$ and use duality to get $(\mathcal{L}_i)_{i<0}$.

Double covers of $\mathbb{P}^1_d$

Theorem (Chan-N. (2021))

Let $d > 2$, let $\mathcal{L}_0 \in E(1,0)$ and let $\mathcal{L}_1 \in E(1,d)$. Then

1. the pair $(\mathcal{L}_0, \mathcal{L}_1)$ extends to a unique helix $\underline{\mathcal{L}}_d$ on Coh$X$
2. cohproj$B_{\underline{\mathcal{L}}_d} \equiv C^{\theta_d}$, where
$$\theta_d = -\frac{2d}{d-2+\sqrt{d^2-4}},$$
3. cohproj$A_{\underline{\mathcal{L}}_d} \equiv \mathbb{P}^1_d$, and
4. the map from Part 2
$$\text{Proj}B_{\underline{\mathcal{L}}_d} \leftrightharpoons \text{Proj}A_{\underline{\mathcal{L}}_d}$$
is a double cover.

Part 5

Second Application: Noncommutative Nonnoetherian $\mathbb{P}^2$'s

Example: Maps from Elliptic Curves to Projective Planes

Embedding of X in $\mathbb{P}^2$

$\mathcal{L}$ = degree 3 line bundle over X, let $V = \text{Hom}(\mathcal{O}_X, \mathcal{L})$.

- $\mathcal{L}$ induces closed immersion

$$X \cong \text{Proj } \Gamma_*(X, \mathcal{L}) \xrightarrow{h} \text{Proj } \mathbb{S}(\Gamma(X, \mathcal{L})) \cong \mathbb{P}^2.$$

- Since $\bigwedge^2 V \cong V^*$, pullback of Koszul complex over $\mathbb{P}^2$ takes form

$$0 \longrightarrow \mathcal{L}^{-3} \longrightarrow V^* \otimes \mathcal{L}^{-2} \longrightarrow V \otimes \mathcal{L}^{-1} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Goal

Look for interesting helices over $C = \text{Coh}X$ with the same "shape" as this example.

Helix construction: main idea

Find sequence $\underline{\mathcal{L}}$ of objects in Coh$X$ with exact sequences like the Koszul complex

$$0 \to \mathcal{L}_{i-3} \to V \otimes \mathcal{L}_{i-2} \to W \otimes \mathcal{L}_{i-1} \to \mathcal{L}_i \to 0$$

Start with *three* simple bundles $(\mathcal{L}_0, \mathcal{L}_1', \mathcal{L}_1)$. Construct a *new* triple $(\mathcal{L}_1, \mathcal{L}_2', \mathcal{L}_2)$ as follows:

$$0 \to \mathcal{L}_0 \to \text{Hom}(\mathcal{L}_0, \mathcal{L}_1)^* \otimes \mathcal{L}_1 \to \text{cok} =: \mathcal{L}_2' \to 0$$

and

$$0 \to \mathcal{L}_1' \to \text{Hom}(\mathcal{L}_1', \mathcal{L}_1)^* \otimes \mathcal{L}_1 \to \text{cok} =: \mathcal{L}_2 \to 0$$

Would like:

- above sequences to be exact,
- $\mathcal{L}_2$, $\mathcal{L}_2'$ simple bundles,
- $\mathcal{L}_1 \to \text{Hom}(\mathcal{L}_1, \mathcal{L}_2)^* \otimes \mathcal{L}_2$ and $\mathcal{L}_2' \to \text{Hom}(\mathcal{L}_2', \mathcal{L}_2)^* \otimes \mathcal{L}_2$ to be injections, etc.

Helix construction: main idea (cont.)

If we can continue to the right, get exact sequences

$$0 \to \mathcal{L}_{i-3} \to \mathrm{Hom}(\mathcal{L}_{i-3}, \mathcal{L}_{i-2})^* \otimes \mathcal{L}_{i-2} \to \mathcal{L}'_{i-1} \to 0$$

and

$$0 \to \mathcal{L}'_{i-1} \to \mathrm{Hom}(\mathcal{L}'_{i-1}, \mathcal{L}_{i-1})^* \otimes \mathcal{L}_{i-1} \to \mathcal{L}_i \to 0$$

which fit together to give

$$0 \to \mathcal{L}_{i-3} \to V \otimes \mathcal{L}_{i-2} \to W \otimes \mathcal{L}_{i-1} \to \mathcal{L}_i \to 0.$$

where

- $V = \mathrm{Hom}(\mathcal{L}_{i-3}, \mathcal{L}_{i-2})^*$,
- $W = \mathrm{Hom}(\mathcal{L}'_{i-1}, \mathcal{L}_{i-1})^* \cong \mathrm{Hom}(\mathcal{L}_{i-1}, \mathcal{L}_i)$.

Definition of $\underline{\mathcal{L}}$

Theorem (Chan-N (2022))

Let $d \geq 3$ be an odd integer. Let $\mathcal{L}_0 \in E(1,0)$, $\mathcal{L}_1' \in E(d,2)$ and let $\mathcal{L}_1 \in E(1,d)$. Then

1. the triple $(\mathcal{L}_0, \mathcal{L}_1', \mathcal{L}_1)$ generates a helix
$$\underline{\mathcal{L}}_d = (\mathcal{L}_i)_{i \in \mathbb{Z}}$$
2. the Koszul complex of $\underline{\mathcal{L}}_d$ is exact of length 3, and
3. helices $\underline{\mathcal{L}}_3$ = Bondal-Polishchuk's *elliptic helices of period* 3 *over* X.

Part 3 $\Rightarrow$ we recover all three-dimensional elliptic Artin-Schelter regular algebras over X when $d = 3$.

Main Theorem

Theorem (Chan-N (2022))

Let $d > 3$ be an odd integer, and let $\mathcal{L}_d$ denote a sequence of the form above. Then

1. $A_{\underline{\mathcal{L}}_d}$ is 3-periodic, Koszul, has global dimension three, and is Gorenstein (with Gorenstein parameter three),
2. $\dim_k(A_{\underline{\mathcal{L}}_d})_{i,i+1} = d$ for all i,
3. $A_{\underline{\mathcal{L}}_d}$ and $B_{\underline{\mathcal{L}}_d}$ are nonnoetherian,
4. the canonical map $\phi : A_{\underline{\mathcal{L}}_d} \longrightarrow B_{\underline{\mathcal{L}}_d}$ is surjective,
5. ϕ induces $\text{Proj} B_{\underline{\mathcal{L}}_d} \leftrightharpoons \text{Proj} A_{\underline{\mathcal{L}}_d}$,
6. $\text{Proj} B_{\underline{\mathcal{L}}_d} \equiv C^{\tau_d}$, i.e. $\text{Proj} B_{\underline{\mathcal{L}}_d}$ is a noncommutative elliptic curve.

Key Tool

Theorem (Chan-N (2022))

If $\mathcal{E}$ and $\mathcal{F}$ are simple bundles such that

- $\mu(\mathcal{E}) < \mu(\mathcal{F})$ and
- $\operatorname{rank} \mathcal{F} < \operatorname{rank} \mathcal{E} \cdot \dim \operatorname{Hom}(\mathcal{E}, \mathcal{F})$,

then evaluation

$$\operatorname{Hom}(\mathcal{E}, \mathcal{F}) \otimes \mathcal{E} \to \mathcal{F}$$

is surjective, and coevaluation

$$\mathcal{F}^* \to \operatorname{Hom}(\mathcal{F}^*, \mathcal{E}^*)^* \otimes \mathcal{E}^*$$

is injective.

Thank You!

A bundle in $E(1,1)$ induces

$$\Phi : E(\gcd(r,d),0) \xrightarrow{\cong} E(r,d).$$

Using Φ, there exists a distinguished bundle $\mathcal{E}_{r,d} \in E(r,d)$ and every bundle in $E(r,d)$ is

$$\mathcal{L} \otimes \mathcal{E}_{r,d}$$

where $\mathcal{L} \in E(1,0)$.

Classification

Let $d > 3$ odd. Let $\mathcal{A}_0 \otimes \mathcal{E}_{1,0} \in E(1,0)$, $\mathcal{A}_1' \otimes \mathcal{E}_{d,2} \in E(d,2)$, $\mathcal{A}_1 \otimes \mathcal{E}_{1,d} \in E(1,d)$.

Question

What are the isomorphism classes of algebras of the form $\mathbb{S}^{nc}(\underline{\mathcal{L}}_d)$?

Rigidify triple by tensoring $\underline{\mathcal{L}}_d$ by $\mathcal{A}_0^*$. Thus, $\mathbb{S}^{nc}(\underline{\mathcal{L}}_d)$ determined by two degree zero line bundles.

Conjecture

The noncommutative symmetric algebra corresponding to $(\mathcal{C}_1, \mathcal{C}_2)$ is isomorphic to that corresponding to $(\mathcal{D}_1, \mathcal{D}_2)$ if and only if $\exists$ an automorphism σ of C^{τ_d} such that $\sigma(\mathcal{C}_i) = \mathcal{D}_i$.