A representation theoretic study of noncommutative symmetric algebras (joint with Daniel Chan)

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Conventions and Notation

- always work over a field k

Part 1

Introduction - The Projective Line

The category coh$\mathbb{P}(V)$

- Let V is 2-diml.
- Define $\mathbb{P}(V) := \text{Proj } \mathbb{S}(V)$.
- Let coh$\mathbb{P}(V)$ be subcategory of Qcoh$\mathbb{P}(V)$ consisting of noetherian objects.

Properties of coh$\mathbb{P}(V)$

- coh$\mathbb{P}(V)$ is hereditary.
- Every object of coh$\mathbb{P}(V)$ is a direct sum of a torsion sheaf and a vector bundle. Every vector bundle is a direct sum of $\mathcal{O}(i)$'s (Grothendieck).

$V := kx \oplus ky$

The Kronecker Algebra $\Lambda(V)$

$$\Lambda(V) = \begin{pmatrix} k & V \\ 0 & k \end{pmatrix}$$

(Right) $\Lambda(V)$-module

$$\longleftrightarrow$$

(N_0, N_1) and $x, y \in \mathrm{Hom}_k(N_0, N_1)$ w/mult

$$(n_0, n_1) \cdot \begin{pmatrix} a & cx + dy \\ 0 & b \end{pmatrix} := (n_0 a, n_0 cx + n_0 dy + n_1 b)$$

Notation: $N_0 \overset{x}{\underset{y}{\rightrightarrows}} N_1$

If a, $b \in k$ not both zero,

$$k \overset{a}{\underset{b}{\rightrightarrows}} k$$

is indecomposable. If $c \neq 0$, then isomorphic to

$$k \overset{ca}{\underset{cb}{\rightrightarrows}} k$$

Heuristic

Points of $\mathbb{P}(V) \to$ Indecomposable $\Lambda(V)$-modules

Beilinson's Theorem

Theorem (Beilinson 1978)

The functor $RHom(\mathcal{O} \oplus \mathcal{O}(1), -)$ gives an equivalence

$$D^b(\mathrm{coh}\mathbb{P}(V)) \to D^b(\mathrm{mod}\Lambda(V)).$$

Regular $\Lambda(V)$-Modules

M is *regular* if it is direct sum of indecomposable $N = (N_0, N_1)$ with $\dim_k(N_0) = \dim_k(N_1)$.

- torsion sheaves over $\mathbb{P}(V) \longleftrightarrow$ regular $\Lambda(V)$-modules
- torsion-free sheaves over $\mathbb{P}(V)$ $= \{\mathcal{N} | \operatorname{Hom}_{\mathbb{P}(V)}(\mathcal{T}, \mathcal{N}) = 0 \text{ for all torsion } \mathcal{T}\}$.

Goal of Talk

Explore the extent to which categories of coherent sheaves over "noncommutative versions" of $\mathbb{P}(V)$ have similar properties, i.e.

- are hereditary,
- have a version of Grothendieck's Theorem for objects, and
- satisfy a version of Beilinson's Theorem.

Part 2

Piontkovski's $\mathbb{P}^1_n$

Coherent rings and modules

A is $\mathbb{N}$-graded algebra, GrA cat. of graded right A-modules.

Definition

Suppose $M \in \text{Gr}A$

- M is **coherent** if M is f.g. and every f.g. submodule is finitely presented.
- A is coherent if it is coherent as a graded right A-module.

Theorem (Chase (1960))

A is coherent iff the full subcategory of GrA of coherent modules is abelian.

If A is coherent, let coh$A$ denote this subcategory of GrA.

Examples/Nonexamples of (graded right) coherence

- A noetherian $\Longrightarrow$ A is coherent.
- $k[\{x_i\}_{i\in\mathbb{N}}]$ is coherent.
- $k\langle x_1, \ldots, x_n\rangle$ is coherent.
- $k\langle x, y, z\rangle/\langle xy, yz, xz - zx\rangle$ is *not* coherent (Polishchuk (2005)).

The category cohproj (Polishchuk (2005))

Let A be coherent connected $\mathbb{N}$-graded algebra and let

- $\text{coh}A$ = cat. of (graded right) coherent modules
- $\text{tors}A$ = full subcat. of right-bounded modules.

Definition

$\text{cohproj}A := \text{coh}A/\text{tors}A$

Remark

If A is noetherian, $\text{cohproj}A \equiv \text{proj}A$.

Theorem (Zhang (1998))

If A is connected, gen. in degree 1 and regular of dim 2 then

$$A \cong k\langle x_1, \ldots, x_n \rangle / \langle b \rangle$$

where $n \geq 2$, $b = \sum_{i=1}^{n} x_i \sigma(x_{n-i+1})$ and $\sigma \in \text{Aut } k\langle x_1, \ldots, x_n \rangle$. Furthermore, A is noetherian iff $n = 2$.

Theorem (Piontkovski (2008))

$n > 2$ implies A is coherent. If $\mathbb{P}^1_n := \text{cohproj}A$, then $\mathbb{P}^1_n$ depends only on n. Furthermore, $\mathbb{P}^1_2 \equiv \text{coh}\mathbb{P}(k^{\oplus 2})$, $\mathbb{P}^1_n$ is hereditary, and there is a Beilinson equivalence

$$D^b(\mathbb{P}^1_n) \equiv D^b(\text{mod}\Lambda(k^{\oplus n})).$$

This form of Beilinson equivalence was independently discovered by Minamoto (2008) and Van den Bergh.

Part 3

Noncommutative Projective Lines

$\mathbb{Z}$-algebras (Bondal and Polishchuk (1993))

The orbit algebra of a sequence

If $\underline{\mathcal{L}} = (\mathcal{L}_i)_{i\in\mathbb{Z}}$ is seq. of objects in a category C, then

$$(A_{\underline{\mathcal{L}}})_{ij} = \mathrm{Hom}(\mathcal{L}_{-j}, \mathcal{L}_{-i})$$

with mult. = composition makes $A_{\underline{\mathcal{L}}} = \oplus_{i,j\in\mathbb{Z}}(A_{\underline{\mathcal{L}}})_{ij}$ a $\mathbb{Z}$-algebra

A ring A is a (positively graded) $\mathbb{Z}$-**algebra** if

- $\exists$ vector space decomp $A = \oplus_{i,j\in\mathbb{Z}} A_{ij}$, with $A_{ij} = 0$ if $j < i$,
- $A_{ij}A_{jk} \subset A_{ik}$,
- $A_{ij}A_{kl} = 0$ for $k \neq j$, and
- the subalgebra A_{ii} contains a unit.

Noncommutative versions of coherent sheaves over $\mathbb{P}^1$: Bimodules

Goal

If V is 2-diml$/k$, $\mathbb{P}^1 = \mathbb{P}(V)$. Idea: replace V by bimod. M.

- D_0, D_1 = division rings over k
- $M = D_0 - D_1$-bimodule of left-right dimension (m, n)

Right dual of M

$M^* := \mathrm{Hom}_{D_1}(M_{D_1}, D_1)$ is $D_1 - D_0$-bimodule with action $(a \cdot \psi \cdot b)(x) = a\psi(bx)$.

Can define ${}^*M = M^{-1*}$ similarly.

Noncommutative versions of coherent sheaves over $\mathbb{P}^1$: Definition (Van den Bergh (2000))

Let M be $D_0 - D_1$-bimodule.

Definition of $\mathbb{S}^{nc}(M)$

- $\exists\ \eta_i : D \to M^{i*} \otimes_D M^{i+1*}$
- $\mathbb{S}^{nc}(M)_{ij} = \frac{M^{i*} \otimes \cdots \otimes M^{j-1*}}{\text{relns. gen. by } \eta_i}$ for $j > i$,
- mult. induced by $\otimes$.

Definition of $\mathbb{P}^{nc}(M)$

Suppose $\mathbb{S}^{nc}(M)$ is coherent. We let

$$\mathbb{P}^{nc}(M) := \text{cohproj}\mathbb{S}^{nc}(M)$$

Motivation for our work

Under what conditions on M is $\mathbb{S}^{nc}(M)$ coherent?

$\mathbb{P}^{nc}(M)$ arises in nature

Examples

- $\mathbb{P}^1_n \equiv \mathbb{P}^{nc}(k^{\oplus n})$ (N (2017)). Implies $\mathbb{P}^1_n = \text{cohproj}\, k\langle x_1, \ldots, x_n \rangle / \langle b \rangle$ independent of choice of b.
- Noncommutative curves of genus zero after Kussin (N (2015))
- Generic fibers of noncommutative ruled surfaces (Patrick (2000), Van den Bergh (2000), D. Chan and N (2016)), noncommutative Del Pezzo surfaces (De Thanhoffer and Presotto (2016)), ruled orders (Artin and de Jong (2005))
- Artin's Conjecture: Every noncommutative surface infinite over its center is birational to some $\mathbb{P}^{nc}(M)$ (1997)

Part 4

Main Theorem

Main Theorem: Preliminary Definitions

Definition

D_0, D_1 division rings (with char $\neq 2$), $M = D_0 - D_1$-bimodule with left-right dimension (m, n). We let

$$\Lambda(M) := \begin{pmatrix} D_0 & M \\ 0 & D_1 \end{pmatrix}$$

Definition

Let M be a $D_0 - D_1$-bimodule. M has *symmetric duals* if M and M^* are finite-dimensional on the left and right and there is a bimodule isomorphism $M \cong M^{**}$.

Main Theorem: Statement

Hypothesis: M has symmetric duals, and the product of its left and right dimensions is ≥ 4.

Theorem (Chan-N (2018))

- $\mathbb{S}^{nc}(M)$ is coherent,
- $\mathbb{P}^{nc}(M)$ is hereditary, and
- There is an equivalence $D^b(\mathbb{P}^{nc}(M)) \equiv D^b(\text{mod}\Lambda(M))$.

Definition

$\mathcal{T} \in \mathbb{P}^{nc}(M)$ is *torsion* if it corresponds to a regular module over $\Lambda(M)$. $\mathcal{N} \in \mathbb{P}^{nc}(M)$ is *torsion-free* if $\text{Hom}_{\mathbb{P}^{nc}(M)}(\mathcal{T}, \mathcal{N}) = 0 \ \forall \ \mathcal{T}$.

Theorem (Chan-N (2018))

Every object of $\mathbb{P}^{nc}(M)$ is a direct sum of a torsion sheaf and a torsion-free sheaf. Every torsion-free sheaf over $\mathbb{P}^{nc}(M)$ is a direct sum of sheaves of the form $e_i\mathbb{S}^{nc}(M)$.

Grothendieck's Theorem

Follows from elementary torsion theory!

Remark

Generalizes results from (N (2014)), (Chan-N (2016)) and simplifies their proofs significantly.

Thank you for your attention!

Main Theorem: Proof

Idea

$\Lambda(M)$ Artinian and hereditary. Work in $\mathrm{mod}\Lambda(M)$.

Key Computation

We compute $\mathrm{RHom}_{D^b(\mathrm{mod}\Lambda(M))}(D(\Lambda(M)), -)$ using bimod right projective resolution of $D(\Lambda(M))$ inspired by (Butler-King (1999)).

Proof sketch:

- **Step 1:** In commutative case, $\mathcal{O}(i) \in \mathrm{coh}\mathbb{P}(k^{\oplus 2})$ corresponds to $\mathcal{L}_i \in D^b(\mathrm{mod}\Lambda(k^{\oplus 2}))$. Now let $\mathcal{N}_i \in D^b(\mathrm{mod}\Lambda(M))$ denote noncommutative analogue of $\mathcal{L}_i$.
- **Step 2:** Prove $\mathbb{S}^{nc}(M) \cong \oplus_{i,j}\mathrm{Hom}_{D^b(\mathrm{mod}\Lambda(M))}(\mathcal{N}_{-j}, \mathcal{N}_{-i})$.
- **Step 3:** Use theorem of Minamoto (2012) to show $\{\mathcal{N}_i\}$ is ample. The fact that $\mathbb{S}^{nc}(M)$ is coherent follows from (Polishchuk (2005)).