Genus zero phenomena in noncommutative algebraic geometry

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Part 1

Introduction

Goal of talk

k=base field

Noncommutative algebraic geometry

Study k-linear abelian categories like coh$X$ where X is a scheme.

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Study k-linear abelian categories like coh$X$ where X is a scheme.

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Warm-up

What are necessary and sufficient conditions on a k-linear abelian category C such that

$$\mathsf{C} \equiv \text{coh}\mathbb{P}^1?$$

Abstract characterization of $\mathbb{P}^1$

Let $\mathsf{C} = \mathrm{coh}\mathbb{P}^1$. Let $\underline{\mathcal{L}} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$. If $\mathcal{L}_i := \mathcal{O}(i)$, then, for all i,

- $\mathrm{End}(\mathcal{L}_i) = k$
- $\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_{i-1}) = \mathrm{Ext}^1(\mathcal{L}_i, \mathcal{L}_{i-1}) = 0$
- $\mathrm{Ext}^1(\mathcal{L}_i, \mathcal{L}_j) = 0 \ \forall j \geq i$
- $\mathrm{Hom}(\mathcal{L}_i, \mathcal{M})$ is f.d. $\forall \mathcal{M} \in \mathsf{C}$
- $\exists$ ses
$$0 \to \mathcal{L}_i \to \mathcal{L}_{i+1}^2 \to \mathcal{L}_{i+2} \to 0$$
whose left map is defined by a basis of $\mathrm{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$.
- $\underline{\mathcal{L}}$ is ample

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Consequence of main theorem

A k-linear abelian category C is equivalent to $\text{coh}\mathbb{P}^1$ iff $\exists$ $\underline{\mathcal{L}} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$ satisfying above properties.

Part 2

Background

Coherent rings and modules

A is $\mathbb{N}$-graded algebra, $\text{Gr}A =$ cat. of graded right A-modules.

Definition

Suppose $M \in \text{Gr}A$

- M is **coherent** if M is f.g. and every f.g. submodule is finitely presented.
- A is coherent if it is coherent as a graded right A-module.

Theorem (Chase (1960))

A is coherent iff the full subcategory of $\text{Gr}A$ of coherent modules is abelian.

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- $k\langle x_1, \ldots, x_n\rangle$ is coherent.
- $k\langle x, y, z\rangle/\langle xy, yz, xz - zx\rangle$ is *not* coherent (Polishchuk (2005)).

$\mathbb{Z}$-algebras (Bondal and Polishchuk (1993))

The orbit algebra of a sequence

If $\underline{\mathcal{L}} = (\mathcal{L}_i)_{i \in \mathbb{Z}}$ is seq. of objects in a category C, then

$$(A_{\underline{\mathcal{L}}})_{ij} = \mathrm{Hom}(\mathcal{L}_{-j}, \mathcal{L}_{-i})$$

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A ring A is a (positively graded) $\mathbb{Z}$-**algebra** if

- $\exists$ vector space decomp $A = \oplus_{i,j \in \mathbb{Z}} A_{ij}$,
- $A_{ij} A_{jk} \subset A_{ik}$,
- $A_{ij} A_{kl} = 0$ for $k \neq j$,
- the subalgebra A_{ii} contains a unit, and
- $A_{ij} = 0$ if $j < i$.

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There is a notion of graded coherence for $\mathbb{Z}$-algebras.

The category cohproj (Polishchuk (2005))

Let A be coherent $\mathbb{Z}$-algebra and let

- coh$A$ = cat. of (graded right) coherent modules
- tors$A$ = full subcat. of right-bounded modules.

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Remark

If A is noetherian, $\mathrm{cohproj} A \equiv \mathrm{proj} A$.

Noncommutative versions of coherent sheaves over $\mathbb{P}^1$: Bimodules

Goal

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- D_0, D_1 = division rings over k
- $M = D_0 - D_1$-bimodule of left-right dimension (m, n)

Right dual of M

$M^* := \mathrm{Hom}_{D_1}(M_{D_1}, D_1)$ is $D_1 - D_0$-bimodule with action $(a \cdot \psi \cdot b)(x) = a\psi(bx)$.

Can define ${}^*M = M^{-1*}$ similarly.

M is **2-periodic** if M^{i*} has left-right dim $= \begin{cases} (m, n) & \text{for } i \text{ even} \\ (n, m) & \text{for } i \text{ odd} \end{cases}$

Noncommutative versions of coherent sheaves over $\mathbb{P}^1$: Definition (Van den Bergh (2000))

Let M be 2-periodic $D_0 - D_1$-bimodule.

Definition of $\mathbb{S}^{nc}(M)$

- $\exists\ \eta_i : D \to M^{i*} \otimes_D M^{i+1*}$
- $\mathbb{S}^{nc}(M)_{ij} = \frac{M^{i*} \otimes \cdots \otimes M^{j-1*}}{\text{relns. gen. by } \eta_i}$ for $j > i$,
- mult. induced by $\otimes$.

Definition of $\mathbb{P}^{nc}(M)$

Suppose $\mathbb{S}^{nc}(M)$ is coherent. We let

$$\mathbb{P}^{nc}(M) := \text{cohproj}\mathbb{S}^{nc}(M)$$

$\mathbb{P}^{nc}(M)$ arises in nature

Examples

- Generic fibers of noncommutative ruled surfaces (Patrick (2000), Van den Bergh (2000), D. Chan and N (2016))
- Generic fibers of noncommutative Del Pezzo surfaces (De Thanhoffer and Presotto (2016))
- Generic fibers of ruled orders (Artin and de Jong (2005))
- Noncommutative curves of genus zero after Kussin (N (2015))
- Artin's Conjecture: Every noncommutative surface infinite over its center is birational to some $\mathbb{P}^{nc}(M)$ (1997)

Theorem (Zhang (1998))

If A is connected, gen. in degree 1 and regular of dim 2 then

$$A \cong k\langle x_1, \dots, x_n\rangle / \langle b\rangle$$

where $n \geq 2$, $b = \sum_{i=1}^{n} x_i \sigma(x_{n-i+1})$ and $\sigma \in \text{Aut } k\langle x_1, \dots, x_n\rangle$.
Furthermore, A is noetherian iff $n = 2$.

Piontkovski's $\mathbb{P}^1_n$

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Theorem (Piontkovski (2008))

$n > 2$ implies A is coherent. If $\mathbb{P}^1_n := \text{cohproj}A$, then $\mathbb{P}^1_n$ depends only on n, is Ext-finite, satisfies Serre duality and is hereditary.

Question

$$\text{Is } \mathbb{P}^1_n \equiv \mathbb{P}^{nc}(M)?$$

Part 3

Results

Recognizing orbit algebras as $\mathbb{S}^{nc}(M)$ (finite-type case)

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- $\text{End}(\mathcal{L}_i) = D_i$ is div. ring f.d./k and $\dim D_i = \dim D_{i+2}$
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- If $l_i := \dim_{D_{i+1}} \text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$, $\exists$ ses

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Theorem N (2017)

If C has sequence $\underline{\mathcal{L}}$ satisfying properties above, then

- ${}_{D_0}M_{D_1} := \mathrm{Hom}(\mathcal{L}_{-1}, \mathcal{L}_0)$ is 2-periodic,
- M does not have type $(1,1)$, $(1,2)$, or $(1,3)$,
- $A_{\underline{\mathcal{L}}} \cong \mathbb{S}^{nc}(M)$.

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Theorem N (2017)

C has $\underline{\mathcal{L}}$ satisfying props iff C $\equiv \mathbb{P}^{nc}(M)$ where

- M is 2-periodic
- M is $D_0 - D_1$-bimodule not of type $(1,1)$, $(1,2)$, $(1,3)$
- $\mathbb{S}^{nc}(M)$ is coherent.

$\mathbb{P}^1$ revisited

An Example

If C $= \text{coh}\mathbb{P}^1$ and $\underline{\mathcal{L}} = (\mathcal{O}(i))_{i \in \mathbb{Z}}$, then $\underline{\mathcal{L}}$ satisfies properties in previous result. Thus,

$$\text{coh}\mathbb{P}^1 \equiv \mathbb{P}^{nc}(V)$$

where V is 2-diml vector-space over k.

$\mathbb{P}^1$ revisited

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where V is 2-diml vector-space over k.

Corollary N (2017)

$$\mathbb{P}^1_n \equiv \mathbb{P}^{nc}(V)$$

where V is n-diml vector space over k.

Explains dependence of $\mathbb{P}^1_n$ on n only.

Thank you for your attention!

More applications

Corollary N (2017)

C has a sequence $\underline{\mathcal{L}}$ satisfying properties above with $D_i = k$ iff C $\equiv \mathbb{P}^1_n$ for some n.

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Corollary N (2017)

If D is nonsplit k-central division ring of deg n, $\exists$ ext. $k \subset k'$ such that

$$\text{Spec } k' \times_{\text{Spec } k} \mathbb{P}^{nc}({}_D D_k) \equiv \mathbb{P}^1_n$$

Generalizes fact that a (commutative) curve of genus zero is a projective line after finite base extension.

Main Theorem: Remark on the general case

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- Finite-dimensionality of $D_i = \text{End}(\mathcal{L}_i)$ can be dropped.
- Need property ensuring if $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})$ has left-right-dimension (m, n) then $\text{Hom}(\mathcal{L}_i, \mathcal{L}_{i+1})^*$ has right-dimension m.