# A CATEGORICAL CHARACTERIZATION OF QUANTUM PROJECTIVE $\mathbb{Z}$-SPACES

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Abstract. In this paper we study a generalization of the notion of AS-regularity for connected $\mathbb{Z}$-algebras defined in [13]. Our main result is a characterization of those categories equivalent to noncommutative projective schemes associated to right coherent regular $\mathbb{Z}$-algebras, which we call quantum projective $\mathbb{Z}$-spaces in this paper.

As an application, we show that smooth quadric hypersurfaces and the standard noncommutative smooth quadric surfaces studied in [23, 15] have right noetherian AS-regular $\mathbb{Z}$-algebras as homogeneous coordinate algebras. In particular, the latter are thus noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$ (in the sense of [26]).

Contents



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Throughout this paper, we work over a field $k$.

## 1. Introduction

In noncommutative algebraic geometry, one studies so-called noncommutative schemes from a geometric perspective. These schemes are often abelian categories with properties in common with categories of coherent sheaves over a scheme. Those noncommutative schemes which behave like categories of coherent sheaves over *projective* schemes have particular significance, as they can be studied via their global invariants. For this reason, it may be useful to characterize these abelian categories, and such a characterization constitutes the main result of this paper.

The first result along these lines was due to Artin and Zhang [2, Theorem 4.5], who characterized those triples $(\mathscr{C}, \mathscr{A}, s)$ consisting of a $k$-linear abelian category $\mathscr{C}$, a distinguished object $\mathscr{A}$ (thought of as a structure sheaf), and an autoequivalence $s$ of the category $\mathscr{C}$, which are equivalent to a triple of the form $(\operatorname{tails} A, \pi(A), s)$, where $A$ is some right noetherian $\mathbb{N}$-graded $k$-algebra satisfying a homological condition called $\chi_1$, $\operatorname{grmod} A$ is the category of finitely generated graded right $A$-modules, $\operatorname{tors} A$ is the full subcategory of $\operatorname{grmod} A$ consisting of right-bounded modules, $\operatorname{tails} A := \operatorname{grmod} A/\operatorname{tors} A$ is the quotient category with quotient functor $\pi$, and (abusing notation) $s$ is induced by the shift functor in $\operatorname{grmod} A$. This result raises a natural question:

**Question 1.1.** Is there a characterization of categories of the form $\operatorname{tails} A$ for suitably well behaved $\mathbb{Z}$-graded algebras $A$?

Artin and Zhang's result was later generalized by the first author and Ueyama [15, Theorem 2.6] to the case in which $A$ is right-coherent. The first author and Ueyama then used this generalization to address Question 1.1. In particular, they obtained a characterization of abelian categories equivalent to noncommutative projective schemes with homogenous coordinate ring a graded right coherent AS-regular algebra over a finite dimensional algebra $R$ of finite global dimension [15, Theorem 4.1].

In a separate development, Bondal and Polishchuk introduced the notion of $\mathbb{Z}$-algebra [3], and illustrated the utility of this concept in the study of $\mathbb{Z}$-graded algebras. Sierra provided further evidence [21] that working with $\mathbb{Z}$-algebras simplifies aspects of the theory of $\mathbb{Z}$-graded algebras. On the other hand, much of the theory of modules over $\mathbb{Z}$-graded algebras can be generalized to the $\mathbb{Z}$-algebra context (see, for example [13]).

Many noncommutative projective schemes with $\mathbb{Z}$-algebra coordinate rings have been studied. For example, Van den Bergh discovered notions of noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$ [26] and noncommutative $\mathbb{P}^1$-bundles over a pair of smooth schemes [27]. Specializing the latter construction to the case where the base schemes are spectra of fields (or division rings), one obtains the notion of a noncommutative projective line. Polishchuk [19] found sufficient conditions for a $k$-linear abelian category to be of the form $\operatorname{tails} A$ for a right coherent positively graded $\mathbb{Z}$-algebra $A$. He applied this result to construct $\mathbb{Z}$-algebra homogenous coordinate rings for noncommutative elliptic curves [18]. Efimov, Lunts and Orlov constructed noncommutative Grassmannians with $\mathbb{Z}$-algebra homogeneous coordinate rings [6], providing further evidence for the significance of the notion of $\mathbb{Z}$-algebra.

Returning to characterizations of noncommutative projective schemes, in [17], the second author characterized those abelian categories equivalent to noncommutative projective lines over a pair of division rings. The purpose of this paper is to obtain a $\mathbb{Z}$-algebra version of [15, Theorem 4.1] which generalizes [17, Theorem 4.2].

Instead of characterizing categories equivalent to $\operatorname{tails} A$ where $A$ is an AS-regular $\mathbb{Z}$-algebra, we use a related notion of regularity, called ASF$^{++}$-regularity (see Section 4). In the $\mathbb{Z}$-graded case, AS-regularity implies that, if $\tau$ is the torsion functor, then $D\,\mathrm{R}\,\tau(A) \cong A(-\ell)_\nu[d]$ in the derived category of graded right $A^e$-modules, for some graded algebra automorphism $\nu \in \operatorname{Aut} A$ called the Nakayama automorphism of $A$. In the $\mathbb{Z}$-algebra case, it is unclear if AS-regularity is enough to guarantee the existence of such an isomorphism, and so we impose it as part of the definition of ASF$^{++}$-regularity.

The following is the main result of the paper, characterizing those noncommutative projective $\mathbb{Z}$-schemes associated to an ASF$^{++}$-regular $\mathbb{Z}$-algebra (Theorem 6.4, Theorem 6.5, and Theorem 6.11), which extends [3, Theorem 4.2], [15, Theorem 4.1], thus providing an answer to the $\mathbb{Z}$-algebra version of Question 1.1. Before we state it, we remark that we will abuse notation in this paper by writing $\mathscr{C} \cong \mathscr{D}$ for categories $\mathscr{C}$ and $\mathscr{D}$ if they are *equivialent* (not necessarily isomorphic) categories.

**Theorem 1.2.** *Let $\mathscr{C}$ be a $k$-linear abelian category. Then $\mathscr{C} \cong \operatorname{tails} C$ for some right coherent ASF$^{++}$-regular $\mathbb{Z}$-algebra $C$ of dimension at least 1 and of Gorenstein parameter $\ell$ if and only if*

(GH1) *$\mathscr{C}$ has a canonical bimodule $\omega_{\mathscr{C}}$, and*
(GH2) *$\mathscr{C}$ has an ample sequence $\{E_i\}_{i\in\mathbb{Z}}$ which is a full geometric helix of period $\ell$ for $\mathscr{D}^b(\mathscr{C})$.*

*In fact, if $\mathscr{C}$ satisfies (GH1) and (GH2), then $C := C(\mathscr{C}, \{E_i\}_{i\in\mathbb{Z}})_{\geq 0}$ is a right coherent ASF$^{++}$-regular $\mathbb{Z}$-algebra of dimension $\operatorname{gldim} \mathscr{C} + 1$ and of Gorenstein parameter $\ell$ such that $\mathscr{C} \cong \operatorname{tails} C$.*

*Moreover, $C$ constructed above is right noetherian if and only if $\mathscr{C}$ is a noetherian category.*

The condition (GH1) requires that $\mathscr{C}$ has an autoequivalence which induces a Serre functor on $\mathscr{D}^b(\mathscr{C})$. The notion of helix we use in (GH2) is similar to that of [3] (see [15, Remark 3.17]). In comparison to [15, Theorem 4.1], Theorem 1.2 is somewhat simpler in that no autoequivalence on $\mathscr{C}$ (other than the canonical bimodule) is required. Our proof of Theorem 1.2 requires the foundations of homological algebra for connected $\mathbb{Z}$-algebras developed in [13] and [14]. In particular, we use a variant of local duality [14, Theorem 2.1] in this paper to prove, in Theorem 6.8, that $\operatorname{tails} C$, where $C$ is ASF$^{++}$-regular, has a Serre functor. Our argument is adapted from [16, Appendix A].

As an application of Theorem 1.2, we construct a family of right noetherian AS-regular $\mathbb{Z}$-algebras from noncommutative quadric hypersurfaces. In particular, we will show that every smooth quadric hypersurface and every standard noncommutative smooth quadric surface has a right noetherian AS-regular $\mathbb{Z}$-algebra as a homogeneous coordinate algebra (Theorem 6.14 and Theorem 6.13).

We now briefly describe the contents of the paper. In Section 2 we recall relevant definitions and results from the theory of $\mathbb{Z}$-algebras we will need. Although some of this material appears in [13], most does not appear elsewhere, and is necessary for defining the notion of an ASF$^{++}$-regular $\mathbb{Z}$-algebra. In Section 3, after recalling the notion of noncommutative projective $\mathbb{Z}$-scheme, we prove variants of the version of local duality from [14] which we will use in the proof of our main theorem. We also include a number of results about derived functors and related triangles which will be used in the sequel.

In Section 4, we continue the study of regularity for $\mathbb{Z}$-algebras initiated in the paper [13]. In [13], we defined two notions of regularity for a $\mathbb{Z}$-algebra, namely, AS-regularity and ASF-regularity. In this paper, after reviewing the aforementioned notions of regularity, we define two more notions of regularity for a $\mathbb{Z}$-algebra, namely ASF$^{+}$-regularity and ASF$^{++}$-regularity, and show the implications:

$$\mathrm{ASF}^{++} \quad \Rightarrow \quad \mathrm{ASF}^{+} \quad \Rightarrow \quad \mathrm{AS} \quad \Rightarrow \quad \mathrm{ASF}$$

(Theorem 4.8, Theorem 4.14, and Theorem 4.19). Recall that AS-regularity and ASF-regularity for a $\mathbb{Z}$-algebra $A$ are the same if $A$ has a "balanced dualizing complex" [13, Theorem 7.10]. In this paper, we show that ASF$^{+}$-regularity and ASF$^{++}$-regularity are the same if $A$ is $\ell$-periodic (Theorem 4.19). We note that the notion of ASF$^{++}$-regularity was first introduced and studied in [14, Definition 3.1].

In Section 5, we introduce several concepts we will need for the proof of Theorem 1.2, including various notions of helix. The proof of Theorem 1.2, as well as some consequences, are given in Section 6. Finally, an application of our main result, to noncommutative quadric hypersurfaces, concludes the paper.

## 2. $\mathbb{Z}$-algebras

Although some of the results in this section are known (see [13]) or easy to see, we will discuss them rather carefully, hoping that this section (together with our previous paper [13]) will serve as a useful reference.

### 2.1. $\mathbb{Z}$-algebras.

A $\mathbb{Z}$-*algebra* is an algebra with vector space decomposition $C = \oplus_{i,j\in\mathbb{Z}} C_{ij}$ and with the multiplication

$$C_{ij} \otimes C_{st} \to \begin{cases} C_{it} & \text{if } j = s \\ 0 & \text{if } j \neq s. \end{cases}$$

A $\mathbb{Z}$-algebra $C$ does not have a unity, but we assume that each subalgebra $C_{ii}$ has a unity $e_i \in C_{ii}$, called a local unity, so that $C_{ij} = e_i C e_j$ (and that $e_i a e_j = a$ for every $a \in C_{ij}$). Let $C, C'$ be $\mathbb{Z}$-algebras. A $\mathbb{Z}$-*algebra homomorphism* $\phi : C \to C'$ is an algebra homomorphism $\phi : C \to C'$ such that $\phi(C_{ij}) \subset C'_{ij}$ for all $i, j \in \mathbb{Z}$, and $\phi(e_i) = e'_i$ for all $i \in \mathbb{Z}$. We say that $C$ is *locally finite* if $\dim_k C_{ij} < \infty$ for all $i, j$, and $C$ is *connected* if $C_{ij} = 0$ for all $i > j$ and $C_{ii} = k$ for all $i$.

Let $C$ be a $\mathbb{Z}$-algebra. A *graded right $C$-module* is a right $C$-module $M = \oplus_{j\in\mathbb{Z}} M_j$ with the action $M_i \otimes C_{ij} \to M_j$. We assume that each $M_i$ is a unitary $C_{ii}$-module in the sense that $me_i = m$ for every $m \in M_i$. The category of graded right $C$-modules is denoted by $\operatorname{GrMod} C$ whose morphisms are right $C$-module homomorphisms preserving degrees. A *graded left $C$-module* is a left $C$-module $M = \oplus_{i\in\mathbb{Z}} M_i$ with the action $C_{ij} \otimes M_j \to M_i$.

Let $C, C'$ be $\mathbb{Z}$-algebras. A *bigraded $C$-$C'$ bimodule* is a $C$-$C'$ bimodule $M = \oplus_{i,j} M_{ij}$ such that $e_i M := \oplus_j M_{ij}$ is a graded right $C'$-module for every $i$ and $Me'_j := \oplus_i M_{ij}$ is a graded left $C$-module for every $j$, that is, we have maps $M_{li} \otimes C'_{ij} \to M_{lj}$ and $C_{ij} \otimes M_{jl} \to M_{il}$. A *homomorphism of bigraded $C$-bimodules* $M, N$ is a homomorphism $\phi : M \to N$ of $C$-bimodules such that $\phi(M_{ij}) \subset N_{ij}$ for every $i, j \in \mathbb{Z}$.

For a graded left $C$-module $M$ and a graded right $C'$-module $N$, $M \otimes_k N := \oplus_{i,j\in\mathbb{Z}} (M_i \otimes_k N_j)$ is naturally a bigraded $C$-$C'$ bimodule. Note that $C$ itself is a bigraded $C$-bimodule. If $C$ is connected, then $C_{\geq n} := \oplus_{j-i\geq n} C_{ij}$ is a bigraded $C$-bimodule for every $n \in \mathbb{N}$.

We define a graded right $C$-module

$$P_i := e_i C = \oplus_{j\in\mathbb{Z}} C_{ij}$$

for every $i \in \mathbb{Z}$, and a graded left $C$-module

$$Q_j := Ce_j = \oplus_{i\in\mathbb{Z}} C_{ij}$$

for every $j \in \mathbb{Z}$. If $C$ is connected, then

$$S_j := e_j(C/C_{\geq 1}) = (C/C_{\geq 1})e_j = P_j e_j = e_j Q_j = e_j C e_j = C_{jj}$$

has a structure of a bigraded $C$-bimodule for every $j \in \mathbb{Z}$. Moreover, $\{P_j\}_{j\in\mathbb{Z}}$ is the set of all indecomposable graded projective right $C$-modules up to isomorphism, and $\{S_j\}_{j\in\mathbb{Z}}$ is the set of all graded simple right $C$-modules up to isomorphism. Note that $S_j$ is the unique simple quotient of $P_j$.

If $M$ is a graded right $C$-module, then $DM = \oplus_{i\in\mathbb{Z}} D(M_i)$ is a graded left $C$-module via $(af)(x) = f(xa)$ where $a \in C_{ij}, x \in M_j, f \in D(M_i) = (DM)_i$ so that $af \in D(M_j) = (DM)_j$. Similarly, if $M$ is a graded left $C$-module, then $DM$ is a graded right $C$-module. If $M$ is a bigraded $C$-$C'$ bimodule, then $DM := \oplus_{i,j} D(M_{ji})$ is naturally a bigraded $C'$-$C$ bimodule. It follows that

$$\begin{aligned} D(e_iM) &= \oplus_j D((e_iM)_j) = \oplus_j D(M_{ij}) = \oplus_j (DM)_{ji} = (DM)e'_i \\ D(Me'_j) &= \oplus_i D((Me'_j)_i) = \oplus_i D(M_{ij}) = \oplus_i (DM)_{ji} = e_j(DM) \end{aligned}$$

for every $i \in \mathbb{Z}$ and every $j \in \mathbb{Z}$, respectively.

We say that $M \in \operatorname{GrMod} C$ is *locally finite* if $\dim_k M_i < \infty$ for every $i$. Note that $C$ is locally finite if and only if $P_j$ is locally finite for every $j$. If $M \in \operatorname{GrMod} C$ is locally finite, then $DDM \cong M$ in $\operatorname{GrMod} C$.

The *opposite $\mathbb{Z}$-algebra of $C$* is the opposite algebra $C^o$ with $C^o_{ij} := C_{-j,-i}$. For $a^o \in C^o_{jk} = C_{-k,-j}, b^o \in C^o_{ij} = C_{-j,-i}$, $b^oa^o = ab \in C_{-k,-i} = C^o_{ik}$, so $C^o$ is in fact a $\mathbb{Z}$-algebra. If $M$ is a graded left $C$-module, then $\oplus_i M_{-i}$ is a graded right $C^o$-module under the action of $a^o \in C^o_{ij} = C_{-j,-i}$ on $x \in M_{-i}$ defined by $xa^o = ax \in M_{-j}$.

In fact, the category of graded left $C$-modules is equivalent to the category of graded right $C^o$-modules, so we often identify these two categories. Similarly, we can see that the category of graded right $C$-modules is equivalent to the category of graded left $C^o$-modules. Both of these facts are proven in [13, Proposition 2.2], where the algebra $C^o$ is denoted $\widetilde{C^{op}}$.

Note that if $C$ is a connected $\mathbb{Z}$-algebra, then $C^o$ is again a connected $\mathbb{Z}$-algebra. Let $C, C'$ be $\mathbb{Z}$-algebras. The category of bigraded $C$-$C'$ bimodules is denoted by $\operatorname{Bimod}(C-C')$. Note that the categories $\operatorname{GrMod} C$ and $\operatorname{Bimod}(C-C')$ are Grothendieck categories [26, Section 3], [13, Proposition 2.2(1)], hence abelian.

*Remark* 2.1. Let $C$ be a $\mathbb{Z}$-algebra.

(1) If $M = \oplus_{i\in\mathbb{Z}} M_i$ is a graded left $C$-module, then, precisely speaking, $\oplus_{i\in\mathbb{Z}} M_{-i} \in \operatorname{GrMod} C^o$ is a graded right $C^o$-module, however, we often identify them in this paper.

(2) If $M$ is a bigraded $C$-bimodule, then $e_iM := \oplus_{j\in\mathbb{Z}} M_{ij}$ is a graded right $C$-module for every $i \in \mathbb{Z}$, and $Me_j := \oplus_{i\in\mathbb{Z}} M_{ij}$ is a graded left $C$-module for every $j \in \mathbb{Z}$, but $Me_j$ is not a graded right $C^o$-module in this grading. By defining $M^o := \oplus_{ij} M_{-j,-i} \in \operatorname{Bimod}(C^o - C^o)$, we identify $Me_j$ with a graded right $C^o$-module $e^o_{-j}M^o := \oplus_{i\in\mathbb{Z}} M^o_{-j,i} := \oplus_{i\in\mathbb{Z}} M_{-i,j} = \oplus_{i\in\mathbb{Z}} (Me_j)_{-i}$.

(3) We write

$$\begin{aligned}
P_i^o &:= e_i^o C^o = \oplus_{j\in\mathbb{Z}} C_{ij}^o = \oplus_{j\in\mathbb{Z}} C_{-j,-i} = Ce_{-i} =: Q_{-i},\\
Q_j^o &:= C^o e_j^o = \oplus_{i\in\mathbb{Z}} C_{ij}^o = \oplus_{i\in\mathbb{Z}} C_{-j,-i} = e_{-j}C =: P_{-j},\\
S_j^o &= e_j^o C^o e_j^o = C_{jj}^o = C_{-j,-j} = e_{-j}Ce_{-j} =: S_{-j}.
\end{aligned}$$

In particular, we identify a graded right $C^o$-module $P_j^o := e_j^o C^o$ with a graded left $C$-module $Q_{-j} := Ce_{-i}$, and so on.

(4) $D : \operatorname{GrMod} C \to \operatorname{GrMod} C^o$ is defined by $D(\oplus_i M_i) := \oplus_i D(M_i)$ when we view $D(\oplus_i M_i)$ as a graded left $C$-module while $D(\oplus_i M_i) := \oplus_i D(M_{-i})$ when we view $D(\oplus_i M_i)$ as a graded right $C^o$-module.

2.2. Hom **and** $\otimes$**.** If $M$ is a bigraded $C'$-$C$ bimodule and $N$ is a graded right $C$-module, then $\underline{\operatorname{Hom}}_C(M,N) := \oplus_i \operatorname{Hom}_C(e_iM,N) = \oplus_i \operatorname{Hom}_C(\oplus_k M_{ik}, N)$ has a structure of a graded right $C'$-module via $(fa)(x) = f(ax)$ where $a \in C'_{\ell i}, x \in \oplus_j M_{ij}, f \in \underline{\operatorname{Hom}}_C(M,N)_\ell = \operatorname{Hom}_C(\oplus_j M_{lj}, N)$ by the map $a\cdot : M_{ij} \to M_{\ell j}$ so that $fa \in \underline{\operatorname{Hom}}_C(M,N)_\ell = \operatorname{Hom}_C(\oplus_j M_{\ell j}, N)$. (Although $\oplus_j M_{ij}$ does not have a structure of a left $C'$-module, the left multiplication $a\cdot : \oplus_j M_{ij} \to \oplus_j M_{\ell j}$ is well-defined, which induces the right action $\cdot a : \operatorname{Hom}_C(\oplus_j M_{ij}, N) \to \operatorname{Hom}_C(\oplus_j M_{\ell j}, N)$.)

If $M$ is a graded right $C$-module and $N$ is a bigraded $C'$-$C$ bimodule, then $\underline{\operatorname{Hom}}_C(M,N) := \oplus_j \operatorname{Hom}_C(M, e_jN) = \oplus_j \operatorname{Hom}_C(M, \oplus_\ell N_{j\ell})$ has a structure of a graded left $C'$-module via $(af)(x) = a(f(x))$ where $a \in C'_{ij}, x \in M, f \in \underline{\operatorname{Hom}}_C(M,N)_j = \operatorname{Hom}_C(M, \oplus_\ell N_{j\ell})$ by the map $a\cdot : N_{j\ell} \to N_{i\ell}$ so that $af \in \underline{\operatorname{Hom}}_C(M,N)_i = \operatorname{Hom}_C(M, \oplus_\ell N_{i\ell})$.

If $M$ is a bigraded $C'$-$C$ bimodule and $N$ is a bigraded $C''$-$C$ bimodule, then $\underline{\operatorname{Hom}}_C(M,N) := \oplus_{i,j} \operatorname{Hom}_C(e_jM, e_iN)$ has a structure of a bigraded $C''$-$C'$ bimodule since

$$\begin{aligned}
e_i'' \underline{\operatorname{Hom}}_C(M,N) &= \oplus_{j\in\mathbb{Z}} \underline{\operatorname{Hom}}_C(M,N)_{ij}\\
&= \oplus_{j\in\mathbb{Z}} \operatorname{Hom}_C(e_j'M, e_i''N)\\
&= \underline{\operatorname{Hom}}_C(M, e_i''N)
\end{aligned}$$

is a graded right $C'$-module for every $i \in \mathbb{Z}$, and

$$\begin{aligned}
\underline{\operatorname{Hom}}_C(M,N)e_j' &= \oplus_{i\in\mathbb{Z}} \underline{\operatorname{Hom}}_C(M,N)_{ij}\\
&= \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_C(e_j'M, e_i''N)\\
&= \underline{\operatorname{Hom}}_C(e_j'M, N)
\end{aligned}$$

is a graded left $C''$-module for every $i \in \mathbb{Z}$.

The proof of the following lemma is straightforward and omitted.

**Lemma 2.2.** *Let $C$ be a $\mathbb{Z}$-algebra.*

(1) *For $M \in \operatorname{GrMod} C$, $\underline{\operatorname{Hom}}_C(C,M) \cong M$ in $\operatorname{GrMod} C$ so that*

$$\operatorname{Hom}_C(P_i, M) = M_i$$

*for every $i \in \mathbb{Z}$. In particular,*

$$\mathrm{Hom}_C(P_i, S_j) = \begin{cases} S_j & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

(2) $\underline{\mathrm{Hom}}_C(P_i, C) \cong Q_i$ *in* $\mathrm{GrMod}\, C^o$*, and* $\underline{\mathrm{Hom}}_{C^o}(Q_i, C^o) \cong P_i$ *in* $\mathrm{GrMod}\, C$ *for every $i \in \mathbb{Z}$.*

*Definition* 2.3. Let $C, C', C''$ be $\mathbb{Z}$-algebras. For $M \in \mathrm{GrMod}\, C$ and $N \in \mathrm{GrMod}\, C^o$, we define

$$M \otimes_C N := \mathrm{Coker}(\oplus_{i,j\in\mathbb{Z}}(M_i \otimes_{C_{ii}} C_{ij} \otimes_{C_{jj}} N_j) \to \oplus_{k\in\mathbb{Z}} M_k \otimes_{C_{kk}} N_k)$$

where the morphism is induced by the usual difference between left and right multiplication.

- For $M \in \mathrm{GrMod}\, C$ and $N \in \mathrm{Bimod}(C - C')$, we define
$$M \underline{\otimes}_C N = \oplus_{\ell\in\mathbb{Z}}(M \otimes_C N e'_\ell) \in \mathrm{GrMod}\, C.$$
- For $M \in \mathrm{Bimod}(C' - C)$ and $N \in \mathrm{GrMod}\, C^o$, we define
$$M \underline{\otimes}_C N = \oplus_{i\in\mathbb{Z}}(e_i M \otimes_C N) \in \mathrm{GrMod}\, C'^o.$$
- For $M \in \mathrm{Bimod}(C - C'), N \in \mathrm{Bimod}(C' - C'')$, we define
$$M \underline{\otimes}_C N = \oplus_{i,j\in\mathbb{Z}}(e_i M \otimes_C N e''_j) \in \mathrm{Bimod}(C - C'').$$

Note that from general properties of adjoint functors and [13, Proposition 5.3(2)], for $N \in \mathrm{Bimod}(C' - C'')$, the functor

$$\underline{\otimes}_C N : \mathrm{Bimod}(C - C') \to \mathrm{Bimod}(C - C'')$$

commutes with colimits.

**Lemma 2.4.** [13, Section 4.1, Proposition 5.3] *Let $C$ be a $\mathbb{Z}$-algebra.*

(1) *For $M \in \mathrm{GrMod}\, C$, $M \underline{\otimes}_C C \cong M$ in $\mathrm{GrMod}\, C$ so that $M \underline{\otimes}_C Q_j \cong M_j$ for every $j \in \mathbb{Z}$.*
(2) *For $N \in \mathrm{GrMod}\, C^o$, $C \underline{\otimes}_C N \cong N$ in $\mathrm{GrMod}\, C^o$ so that $P_i \underline{\otimes}_C N \cong N_i$ for every $i \in \mathbb{Z}$.*
(3) *For $M \in \mathrm{GrMod}\, C, N \in \mathrm{GrMod}\, C'$ and $L \in \mathrm{Bimod}(C - C')$,*
$$\mathrm{Hom}_{C'}(M \underline{\otimes}_C L, N) \cong \mathrm{Hom}_C(M, \underline{\mathrm{Hom}}_{C'}(L, N)).$$

2.3. **Noetherian and Coherent Properties.** For a set of objects $\mathcal{E}$ in an additive category $\mathscr{C}$, we denote by $\mathrm{add}\, \mathcal{E}$ the set of objects in $\mathscr{C}$ consisting of all finite direct sums of objects in $\mathcal{E}$.

*Remark* 2.5. The notation $\mathrm{add}\, \mathcal{E}$ usually denotes the set of objects in $\mathscr{C}$ consisting of direct summands of finite direct sums of objects in $\mathcal{E}$, so the above notation is not standard.

*Definition* 2.6. Let $C$ be a $\mathbb{Z}$-algebra.

(1) We say that $M \in \mathrm{GrMod}\, C$ is *finitely generated* (resp. *finitely presented*) if there exists an exact sequence $F^0 \to M \to 0$ (resp. $F^1 \to F^0 \to M \to 0$) where $F^i \in \mathrm{add}\{P_j\}_{j\in\mathbb{Z}}$.

(2) We say that $M \in \operatorname{GrMod} C$ is *coherent* if $M$ is finitely generated and $\operatorname{Ker} \phi$ is finitely generated for every homomorphism $\phi : F \to M$ with $F \in \operatorname{add}\{P_j\}_{j \in \mathbb{Z}}$.
(3) We denote by $\operatorname{grmod} C$ the full subcategory of $\operatorname{GrMod} C$ consisting of finitely presented modules, and by $\operatorname{coh} C$ the full subcategory of $\operatorname{GrMod} C$ consisting of coherent modules.
(4) We say that $C$ is *right coherent* if $P_i, S_i \in \operatorname{coh} C$ for every $i \in \mathbb{Z}$.

We call a module in $\operatorname{add}\{P_j\}_{j \in \mathbb{Z}}$ *finitely generated free*. In this terminology, $C = \oplus_j P_j \in \operatorname{GrMod} C$ itself is free but not finitely generated free. If $C$ is connected, then every finitely generated projective graded right $C$-module is isomorphic to a module in $\operatorname{add}\{P_j\}_{j \in \mathbb{Z}}$.

The following result will often be used without comment in the sequel.

**Lemma 2.7.** *Let $C$ be a locally finite connected $\mathbb{Z}$-algebra.*
(1) $\operatorname{coh} C$ *is an abelian category.*
(2) *If $C$ is right coherent, then* $\operatorname{grmod} C = \operatorname{coh} C$ *so that* $\operatorname{grmod} C$ *is an abelian category.*
(3) *Conversely, if* $\operatorname{grmod} C$ *is an abelian category, then $P_j \in \operatorname{coh} C$ for every $j \in \mathbb{Z}$.*

*Proof.* (1) This follows from [19, Proposition 1.1].

(2) If $M \in \operatorname{coh} C$, then $M \in \operatorname{grmod} C$ by definition. Conversely, if $C$ is right coherent and $M \in \operatorname{grmod} C$, then there exists an exact sequence $F^1 \to F^0 \to M \to 0$ in $\operatorname{GrMod} C$ where $F^1, F^0 \in \operatorname{add}\{P_j\}_{j \in \mathbb{Z}} \subset \operatorname{coh} C$. Since $\operatorname{coh} C$ is an abelian category by (1), $M \cong \operatorname{Coker}(F^1 \to F^0) \in \operatorname{coh} C$.

(3) If $\operatorname{grmod} C$ is an abelian category, then, for every homomorphism $\phi : F \to P_j$ with $F, P_j \in \operatorname{add}\{P_j\}_{j \in \mathbb{Z}} \subset \operatorname{grmod} C$, $\operatorname{Ker} \phi \in \operatorname{grmod} C$. In particular, $P_j, \operatorname{Ker} \phi$ are finitely generated, so $P_j \in \operatorname{coh} C$. □

*Definition* 2.8. We say that a $\mathbb{Z}$-algebra $C$ is right noetherian if $P_j \in \operatorname{GrMod} C$ is a noetherian object for every $j \in \mathbb{Z}$.

By [26, Definition 3.1], $C$ is right noetherian if and only if $\operatorname{GrMod} C$ is a locally noetherian (Grothendieck) catetgory.

**Lemma 2.9.** *If $C$ is a right noetherian $\mathbb{Z}$-algebra, then every noetherian module is coherent. In particular, $C$ is right coherent and* $\operatorname{grmod} C$ *is an abelian category.*

*Proof.* If $M \in \operatorname{GrMod} C$ is a noetherian module, then $M$ is finitely generated, so there exists $F \in \operatorname{add}\{P_j\}_{j \in \mathbb{Z}}$ and a surjection $\phi : F \to M$. Since $P_j$ is noetherian for every $j \in \mathbb{Z}$, $F$ is noetherian, so $\operatorname{Ker} \phi \subset F$ is noetherian. It follows that $\operatorname{Ker} \phi$ is finitely generated, so $M \in \operatorname{coh} C$. In particular, since $P_j, S_j \in \operatorname{GrMod} C$ are noetherian for every $j \in \mathbb{Z}$, $P_j, S_j \in \operatorname{coh} C$, so $C$ is right coherent. □

**Lemma 2.10.** *A $\mathbb{Z}$-algebra $C$ is right noetherian if and only if* $\operatorname{grmod} C$ *is a noetherian category.*

*Proof.* For every $M \in \operatorname{grmod} C$, there exists a surjection $F \to M$ in $\operatorname{GrMod} C$ where $F \in \operatorname{add}\{P_j\}_{j\in\mathbb{Z}} \subset \operatorname{GrMod} C$ is a noetherian object, so $M \in \operatorname{GrMod} C$ is a noetherian object, hence $\operatorname{grmod} C$ is a noetherian category.

Conversely, if $\operatorname{grmod} C$ is a noetherian category, then $P_j \in \operatorname{grmod} C$ is a noetherian object for every $j \in \mathbb{Z}$, so $C$ is right noetherian. $\square$

2.4. **Module Categories.** Let $C$ be a $\mathbb{Z}$-algebra. We denote by $\operatorname{Mod} C$ the category of right $C$-modules which is "unitary" in the sense that $M = MC$. The following lemma is known (cf. [21], [26]). We give a proof for the convenience of the reader.

**Lemma 2.11.** *For every $\mathbb{Z}$-algebra $C$, $\operatorname{GrMod} C \cong \operatorname{Mod} C$.*

*Proof.* Let $M \in \operatorname{Mod} C$. Since $e_ie_j = \begin{cases} e_j & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$ if $me_i = ne_j \in Me_i \cap Me_j$ for $i \neq j$, then $ne_j = ne_j^2 = me_ie_j = 0$, so $\oplus_{j\in\mathbb{Z}} Me_j \subset M$. Since $M = MC$, for every $m \in M$, there exist $n \in M$ and $a = \sum_{i,j} a_{ij} \in \oplus_{i,j\in\mathbb{Z}} C_{ij} = C$ such that $m = na = \sum_{i,j} na_{ij} = \sum_j (\sum_i na_{ij})e_j \in \oplus_{j\in\mathbb{Z}} Me_j$, so $M = \oplus_{j\in\mathbb{Z}} Me_j$. It is easy to see that $\oplus_{i\in\mathbb{Z}} Me_i$ is naturally a graded right $C$-module. If $\phi : M \to N$ is a homomorphism of right $C$-modules, then $\phi(me_j) = \phi(m)e_j$, so $\phi(Me_j) \subset Ne_j$, hence $\phi$ is naturally a homomorphism of graded right $C$-modules. It follows that $\operatorname{Mod} C \to \operatorname{GrMod} C$; $M \mapsto \oplus_{j\in\mathbb{Z}} Me_j$ is an equivalence functor. $\square$

*Remark* 2.12. By Lemma 2.11, we have the following:

1. For a $\mathbb{Z}$-algebra $C$ and $M, N \in \operatorname{GrMod} C$, $M \cong N$ in $\operatorname{GrMod} C$ if and only if $M \cong N$ in $\operatorname{Mod} C$.
2. For $\mathbb{Z}$-algebras $C, C'$, $\operatorname{GrMod} C \cong \operatorname{GrMod} C'$ if and only if $\operatorname{Mod} C \cong \operatorname{Mod} C'$.

**Lemma 2.13.** *Let $C, C'$ be $\mathbb{Z}$-algebras. If $C \cong C'$ as algebras (not necessarily as $\mathbb{Z}$-algebras), then $\operatorname{GrMod} C \cong \operatorname{GrMod} C'$.*

*Proof.* Let $\phi : C \to C'$ be an isomorphism of algebras. If $M \in \operatorname{Mod} C'$, then $MC' = M$, so $M_\phi C := M\phi(C) = MC' = M$, hence $M_\phi \in \operatorname{Mod} C$. It follows that $\operatorname{GrMod} C \cong \operatorname{Mod} C \cong \operatorname{Mod} C' \cong \operatorname{GrMod} C'$ by Lemma 2.11. $\square$

2.5. **Periodicity.** Let $C, C'$ be $\mathbb{Z}$-algebras, and $\phi : C \to C'$ a homomorphism of $\mathbb{Z}$-algebras. For $M \in \operatorname{GrMod} C'$, we define $M_\phi \in \operatorname{GrMod} C$ by $M_\phi = M$ as a graded vector space with the action $m * a = m\phi(a)$ for $m \in M, a \in C$. This induces a functor $(-)_\phi : \operatorname{GrMod} C' \to \operatorname{GrMod} C$.

For a bigraded vector space $M$ and $r, \ell \in \mathbb{Z}$, we define a bigraded vector space $M(r, \ell)$ by $M(r, \ell)_{ij} = M_{r+i,\ell+j}$.

**Lemma 2.14.** *Let $C, C'$ be $\mathbb{Z}$-algebras.*

1. *For $\ell \in \mathbb{Z}$, $C(\ell) := C(\ell, \ell)$ is a $\mathbb{Z}$-algebra.*
2. *If $M$ is a bigraded $C$-$C'$ bimodule, then $M(\ell, r)$ is a bigraded $C(\ell)$-$C'(r)$ bimodule.*

(3) *If $\phi : C \to C(-\ell)$ is an isomorphism of $\mathbb{Z}$-algebras, then $\phi^{-1}$ induces an isomorphism $C \to C(\ell)$ of $\mathbb{Z}$-algebras, which is denoted by $\phi^{-1}$ by abuse of notation, and $C(0,-\ell)_\phi \cong {}_{\phi^{-1}}C(\ell,0)$ as bigraded $C$-bimodules.*

*Proof.* (1) Note that $C = C(\ell)$ as ungraded algebras. Since

$$C(\ell)_{ij} \otimes C(\ell)_{jk} = C_{i+\ell,j+\ell} \otimes C_{j+\ell,k+\ell} \to C_{i+\ell,k+\ell} = C(\ell)_{ik},$$

the result follows.

(2) Note that $M(\ell, r) = M$ as ungraded $C$-$D$ bimodules. Since

$$C(\ell)_{ij} \otimes M(\ell,r)_{jk} = C_{\ell+i,\ell+j} \otimes M_{\ell+j,r+k} \to M_{\ell+i,r+k} = M(\ell,r)_{ik},$$
$$M(\ell,r)_{ij} \otimes D(r)_{jk} = M_{\ell+i,r+j} \otimes D_{r+j.r+k} \to M_{\ell+i,r+k} = M(\ell,r)_{ik},$$

the result follows.

(3) Since $\phi^{-1} : C \to C(\ell)$ is an isomorphism of (ungraded) algebras, and

$$\phi^{-1}(C_{ij}) = \phi^{-1}(C(-\ell)_{i+\ell,j+\ell}) = C_{i+\ell,j+\ell} = C(\ell)_{ij},$$

$\phi^{-1} : C \to C(\ell)$ is an isomorphism of $\mathbb{Z}$-algebras.

Since $\phi : {}_{\phi^{-1}}C(\ell,0) \to C(0,-\ell)_\phi$ is an isomorphism of bigraded vector spaces, and

$$\phi(a * mb) = \phi(\phi^{-1}(a)mb) = a\phi(m)\phi(b) = a\phi(m) * b,$$

for $m \in {}_{\phi^{-1}}C(0,-\ell)$ and $a, b \in C$, $\phi : {}_{\phi^{-1}}C(\ell,0) \to C(0,-\ell)_\phi$ is an isomorphism of bigraded $C$-bimodules. $\square$

Let $C$ be a $\mathbb{Z}$-algebra. Since $C \cong C(r)$ as algebras (but not necessarily as $\mathbb{Z}$-algebras in general), $\operatorname{GrMod} C \cong \operatorname{GrMod} C(r)$ by Lemma 2.13. In fact, we have an equivalence functor $(r) : \operatorname{GrMod} C \to \operatorname{GrMod} C(r)$ defined by $M(r) := \oplus_{j\in\mathbb{Z}} M_{r+j}$. Since $(e_iC)(r) = \oplus_{j\in\mathbb{Z}} C_{i,j+r} = e_{i-r}(C(r))$, the assignment $F \mapsto F(r)$ preserves finitely generated projectives, so $(r)$ induces an equivalence functor $(r) : \operatorname{grmod} C \to \operatorname{grmod} C(r)$. For $M \in \operatorname{GrMod} C$, $M(r)$ is not a graded right $C$-module in general, so there exists no notion of homomorphism of graded right $C$-modules of degree $r$.

**Lemma 2.15.** *Let $C, C', C''$ be $\mathbb{Z}$-algebras.*

(1) *For $\ell \in \mathbb{Z}$, $-\underline{\otimes}_C C(0,\ell) \cong (-)(\ell) : \operatorname{GrMod} C \to \operatorname{GrMod} C(\ell)$ as functors.*
(2) *If $\phi : C'' \to C'$ is a homomorphism of $\mathbb{Z}$-algebras and $M$ is a bigraded $C$-$C'$ bimodule, then $-\underline{\otimes}_C M_\phi \cong (-\underline{\otimes}_C M)_\phi : \operatorname{GrMod} C \to \operatorname{GrMod} C''$ as functors.*
(3) *In particular, if $\phi : C \to C(\ell)$ is a homomorphism of $\mathbb{Z}$-algebras, then $- \otimes_C C(0,\ell)_\phi \cong (-)(\ell)_\phi : \operatorname{GrMod} C \to \operatorname{GrMod} C$ as functors.*

*Proof.* Functors of both sides are naturally isomorphic on ungraded module categories and they are compatible with the grading, hence the result. $\square$

We say that $C$ is *$r$-periodic* if $C(r) \cong C$ as $\mathbb{Z}$-algebras. If $C$ is $r$-periodic and $\phi : C \to C(r)$ is an isomorphism of $\mathbb{Z}$-algebras, then, for $M \in \operatorname{GrMod} C$,

$M(r)_\phi \in \operatorname{GrMod} C$, so there exist autoequivalences of $\operatorname{GrMod} C$ and $\operatorname{grmod} C$ defined by $M \mapsto M(r)_\phi$.

**Lemma 2.16.** *If $C$ is an $r$-periodic $\mathbb{Z}$-algebra, then an isomorphism $\phi : C \to C(r)$ of $\mathbb{Z}$-algebras restricts to an isomorphism $P_\ell \to P_{\ell+r}(r)_\phi$ in $\operatorname{GrMod} C$ for every $\ell \in \mathbb{Z}$.*

*Proof.* If $\phi : C \to C(r)$ is an isomorphism of $\mathbb{Z}$-algebras, then $P_{\ell+r}(r)_\phi = \oplus_i C_{\ell+r,i+r} = \phi(P_\ell)$ as graded vector spaces. For $a \in P_{\ell+r}(r)_\phi$ and $b \in C$, $\phi(a) * b = \phi(a)\phi(b) = \phi(ab)$, so we have a commutative diagram

$$\begin{array}{ccccc}
(P_\ell)_i \otimes C_{ij} = & C_{\ell i} \otimes C_{ij} & \xrightarrow{\cdot} & C_{\ell j} & = (P_\ell)_j \\
& {\scriptstyle \phi\otimes\mathrm{id}}\downarrow & & \downarrow{\scriptstyle \phi} & \\
P_{\ell+r}(r)_i \otimes C_{ij} = & C_{\ell+r,i+r} \otimes C_{ij} & \xrightarrow{*} & C_{\ell+r,j+r} & = P_{\ell+r}(r)_j
\end{array}$$

hence $\phi : P_\ell \to P_{\ell+r}(r)_\phi$ is an isomorphism in $\operatorname{GrMod} C$ for every $\ell \in \mathbb{Z}$. $\square$

2.6. **$\mathbb{Z}$-algebras Associated to Graded Algebras.** For a graded algebra $A$, we define a $\mathbb{Z}$-algebra $\overline{A}$ by $\overline{A} := \oplus_{i,j\in\mathbb{Z}} A_{j-i}$. The following lemma is well-known (cf. [3], [19], [21]).

**Lemma 2.17.** *For a graded algebra $A$, the functors $\Phi : \operatorname{GrMod} A \to \operatorname{GrMod} \overline{A}$ and $\Phi : \operatorname{grmod} A \to \operatorname{grmod} \overline{A}$ defined by $\Phi(M) := \oplus_{i\in\mathbb{Z}} M_i$ are equivalences of categories sending $A(-i)$ to $P_i$ and $k(-i)$ to $S_i$ for every $i \in \mathbb{Z}$.*

*Proof.* It is well-known that the functor $\Phi : \operatorname{GrMod} A \to \operatorname{GrMod} \overline{A}$ defined as above is an equivalence functor (cf. [21]). Since

$$\begin{aligned}
\Phi(A(-i)) &:= \oplus_j A(-i)_j = \oplus_j A_{j-i} = \oplus_j \overline{A}_{ij} = e_i\overline{A} =: P_i, \\
\Phi(k(-i)) &:= \oplus_j k(-i)_j = \oplus_j k_{j-i} = e_i\overline{A}e_i =: S_i,
\end{aligned}$$

$\Phi$ restricts to an equivalence functor $\Phi : \operatorname{grmod} A \to \operatorname{grmod} \overline{A}$. $\square$

**Lemma 2.18.** *Let $A$ be a graded algebra. Viewing $\operatorname{GrMod} \overline{A}^o$ as the category of graded left $\overline{A}$-modules, the following hold.*

1. *$\Phi^o : \operatorname{GrMod} A^o \to \operatorname{GrMod} \overline{A}^o$ defined by $\Phi^o(M) := \oplus_{j\in\mathbb{Z}} M_{-j}$ is an equivalence functor such that $\Phi^o(A(i)) \cong \overline{A}e_i$ for every $i \in \mathbb{Z}$.*
2. *$\operatorname{GrMod} A^e \to \operatorname{Bimod}(\overline{A} - \overline{A})$; $M \mapsto \overline{M} := \oplus_{i,j\in\mathbb{Z}} M_{j-i}$ is a functor such that, for every $M \in \operatorname{GrMod} A^e$, $\Phi(M(-i)) \cong e_i\overline{M}$ for every $i \in \mathbb{Z}$, and $\Phi^o(M(i)) \cong \overline{M}e_{-i}$ for every $i \in \mathbb{Z}$. (Here we view $\Phi : \operatorname{GrMod} A^e \to \operatorname{GrMod} A \to \operatorname{GrMod} \overline{A}$ and $\Phi^o : \operatorname{GrMod} A^e \to \operatorname{GrMod} A^o \to \operatorname{GrMod} \overline{A}^o$ by composing with the natural functors $\operatorname{GrMod} A^e \to \operatorname{GrMod} A$ forgetting the graded left $A$-module structure and $\operatorname{GrMod} A^e \to \operatorname{GrMod} A^o$ forgetting the graded right $A$-module structure.)*
3. *$\Phi^o D = D\Phi : \operatorname{GrMod} A \to \operatorname{GrMod} \overline{A}^o$ as functors, and $\Phi D = D\Phi^o : \operatorname{GrMod} A^o \to \operatorname{GrMod} \overline{A}$ as functors.*

*Proof.* (1) By Lemma 2.17, $\Phi^o : \operatorname{GrMod} A^o \to \operatorname{GrMod} \overline{A}^o$ defined by $\Phi^o(M) := \oplus_{j\in\mathbb{Z}} M_j$ is an equivalence functor such that $\Phi^o(A^o(-i)) \cong e_i^o\overline{A}^o$ for every $i \in \mathbb{Z}$ viewed as a graded right $\overline{A}^o$-module, however, $\Phi^o(M) = \oplus_{j\in\mathbb{Z}} M_{-j}$ and $e_i^o\overline{A}^o = \overline{A}e_{-i}$ viewed as a graded left $\overline{A}$-module by Remark 2.1.

(2) Let $M \in \operatorname{GrMod} A^e$. For $x \in \overline{M}_{ij} = M_{j-i}, a \in \overline{A}_{si} = A_{i-s}, b \in \overline{A}_{jt} = A_{t-j}$, $axb \in M_{t-s} = \overline{M}_{st}$, so we may view $\overline{M} \in \operatorname{Bimod}(\overline{A} - \overline{A})$.

Let $\phi \in \operatorname{Hom}_{A^e}(M, N)$ where $M, N \in \operatorname{GrMod} A^e$. For $x \in \overline{M}_{ij} = M_{j-i}$, $\phi(x) \in N_{j-i} = \overline{N}_{ij}$, so we may define a map $\overline{\phi} : \overline{M} \to \overline{N}$ by $\overline{\phi}(x) := \phi(x)$ such that $\phi(\overline{M}_{ij}) \subset \overline{N}_{ij}$. Since $\overline{\phi}(axb) = \phi(axb) = a\phi(x)b = a\overline{\phi}(x)b$, $\overline{\phi} \in \operatorname{Hom}_{\overline{A}}(\overline{M}, \overline{N})$. Moreover, we have

$$\Phi(M(-i)) := \oplus_{j\in\mathbb{Z}} M(-i)_j = \oplus_{j\in\mathbb{Z}} M_{j-i} = \oplus_{j\in\mathbb{Z}} \overline{M}_{ij} =: e_i\overline{M},$$

and

$$\Phi^o(M(i)) := \oplus_{j\in\mathbb{Z}} M(i)_{-j} = \oplus_{j\in\mathbb{Z}} M_{i-j} = \oplus_{j\in\mathbb{Z}} \overline{M}_{-j,-i} =: \overline{M}e_{-i}.$$

(3) Recall that $D : \operatorname{GrMod} A \to \operatorname{GrMod} A^o$ is defined by $D(\oplus_i M_i) := \oplus_i D(M_{-i})$ while $D : \operatorname{GrMod} \overline{A} \to \operatorname{GrMod} \overline{A}^o$ is defined by $D(\oplus_i M_i) := \oplus_i D(M_{-i})$ when we view $D(\oplus_i M_i)$ as a graded left $\overline{A}$-module (see Remark 2.1 (4)). Since

$$\Phi^o D(\oplus_i M_i) = \Phi^o(\oplus_i D(M_{-i})) = \oplus_i D(M_i) = D(\oplus_i M_i) = D\Phi(\oplus_i M_i)$$

in $\operatorname{GrMod} \overline{A}^o$ for $M \in \operatorname{GrMod} A$, we see that $\Phi^o D = D\Phi : \operatorname{GrMod} A \to \operatorname{GrMod} \overline{A}^o$.

Similarly, since

$$\Phi D(\oplus_i M_i) = \Phi(\oplus_i D(M_{-i})) = \oplus_i DM_{-i} = D(\oplus_i M_{-i}) = D\Phi^o(\oplus_i M_i)$$

in $\operatorname{GrMod} \overline{A}$ for $M \in \operatorname{GrMod} A^o$, we see that $\Phi D = D\Phi^o : \operatorname{GrMod} A^o \to \operatorname{GrMod} \overline{A}$. $\square$

*Remark* 2.19. For a graded algebra $A$, $\overline{(A^o)}_{ij} = {A^o}_{j-i} = A_{-i+j} = \overline{A}_{-j,-i} = {(\overline{A})^o}_{ij}$ for every $i, j \in \mathbb{Z}$, so $\overline{(A^o)} = (\overline{A})^o$ as $\mathbb{Z}$-algebras. In particular, $A$ is a connected graded algebra if and only if $A^o$ is a connected graded algebra if and only if $\overline{A}$ is a connected $\mathbb{Z}$-algebra if and only if $(\overline{A})^o$ is a connected $\mathbb{Z}$-algebra.

**Lemma 2.20.** [21, Proposition 3.1] *Let $C$ be a $\mathbb{Z}$-algebra. Then there exists a graded algebra $A$ such that $C \cong \overline{A}$ as $\mathbb{Z}$-algebras if and only if $C$ is 1-periodic.*

## 3. Derived Categories of Graded Modules

One of the main results in [13] and [14] is a version of local duality for connected $\mathbb{Z}$-algebras. In this section, we prove another version of local duality for $\mathbb{Z}$-algebras (Theorem 3.29) and we prove other results about derived functors associated to noncommutative projective $\mathbb{Z}$-schemes which will be employed in the sequel.

Throughout the remainder of the paper, if $\mathscr{C}$ denotes an abelian category, we let $\mathscr{D}(\mathscr{C})$ (resp. $\mathscr{D}^-(\mathscr{C})$, $\mathscr{D}^+(\mathscr{C})$, $\mathscr{D}^b(\mathscr{C})$) denote the derived category of $\mathscr{C}$ (resp. the bounded above derived category of $\mathscr{C}$, the bounded below derived category, the bounded derived category). We will also utilize various left and right derived functors of functors already introduced, and the reader may consult [13, Section 6] for more information about these derived functors.

### 3.1. **Noncommutative Projective $\mathbb{Z}$-schemes.**

*Definition* 3.1. Let $C$ be a $\mathbb{Z}$-algebra and $X \in \mathscr{D}^b(\operatorname{GrMod} C)$. A complex $(F, d)$

$$\cdots \xrightarrow{d^2} F^2 \xrightarrow{d^1} F^1 \xrightarrow{d^0} F^0 \longrightarrow M \longrightarrow 0$$

where $F^q \in \operatorname{add}\{P_j\}_{j\in\mathbb{Z}}$ is called a *finitely generated free resolution of* $X$ if $F$ is quasi-isomorphic to $X$. A finitely generated free resolution of $X$ is called *minimal* if $\operatorname{Im} d^q \subset F^q C_{\geq 1}$ for every $q \in \mathbb{Z}$.

**Lemma 3.2.** *If $C$ is a right coherent connected $\mathbb{Z}$-algebra, then every $M \in \operatorname{grmod} C$ has a unique minimal finitely generated free resolution up to isomorphism.*

*Proof.* The existence of a minimal finitely generated free resolution follows from [13, Proposition 4.4]. Using the same argument as in the connected graded case, a minimal finitely generated free resolution over a connected $\mathbb{Z}$-algebra is unique up to isomorphism. □

The following condition is essential for the rest of the paper.

*Definition* 3.3. A connected $\mathbb{Z}$-algebra $C$ is called *right* Ext-*finite* if every $S_i$ has a minimal finitely generated free resolution in $\operatorname{GrMod} C$.

**Lemma 3.4.** *If $C$ is a right coherent connected $\mathbb{Z}$-algebra, then $C$ is right* Ext-*finite.*

*Proof.* This follows from Lemma 3.2. □

**Lemma 3.5.** *Let $C$ be a connected $\mathbb{Z}$-algebra. If $C$ is right* Ext-*finite, then $C$ is locally finite. In particular, if $C$ is right coherent, then $C$ is locally finite.*

*Proof.* This follows from [13, Remark 3.3]. □

In summary, we have the following implications for a connected $\mathbb{Z}$-algebra.

right noetherian ⇒ right coherent ⇒ right Ext-finite ⇒ locally finite.

*Definition* 3.6. Let $C$ be a $\mathbb{Z}$-algebra. We say that $M \in \operatorname{GrMod} C$ is *right bounded* if $M_{\geq m} = 0$ for some $m \in \mathbb{Z}$ and *left bounded* if $M_{\leq m} = 0$ for some $m \in \mathbb{Z}$.

We denote by $\operatorname{Tors} C$ the full subcategory of $\operatorname{GrMod} C$ consisting of modules $M$ such that $xC$ is right bounded for every $x \in M$.

We define a torsion functor $\tau : \operatorname{GrMod} C \to \operatorname{Tors} C \subset \operatorname{GrMod} C$ by

$$\tau(M) := \{x \in M \mid xC \text{ is right bounded}\}.$$

**Lemma 3.7.** *If $C$ is a right* Ext-*finite connected $\mathbb{Z}$-algebra, then $\operatorname{Tors} C$ is a localizing subcategory of $\operatorname{GrMod} C$.*

*Proof.* By [13, Lemma 3.5], $\operatorname{Tors} C$ is a Serre subcategory of $\operatorname{GrMod} C$. Since $\tau(M)$ is the largest torsion submodule of $M \in \operatorname{GrMod} C$, $\operatorname{Tors} C$ is a localizing subcategory of $\operatorname{GrMod} C$ by [20, Proposition 4.5.2]. □

*Definition* 3.8. Let $C$ be a right Ext-finite connected $\mathbb{Z}$-algebra. We define the quotient category $\operatorname{Tails} C := \operatorname{GrMod} C/\operatorname{Tors} C$. We call $\operatorname{Tails} C$ the *noncommutative projective $\mathbb{Z}$-scheme associated to $C$.*

Let $C$ be a right Ext-finite connected $\mathbb{Z}$-algebra. We denote by

$$\pi : \operatorname{GrMod} C \to \operatorname{Tails} C$$

the quotient functor. Since $\operatorname{Tors} C$ is a localizing subcategory of $\operatorname{GrMod} C$ by Lemma 3.7, $\pi : \operatorname{GrMod} C \to \operatorname{Tails} C$ has a right adjoint

$$\omega : \operatorname{Tails} C \to \operatorname{GrMod} C.$$

We write

$$Q := \omega\pi : \operatorname{GrMod} C \to \operatorname{GrMod} C.$$

We often write $\mathcal{M} := \pi M \in \operatorname{Tails} C$ for $M \in \operatorname{GrMod} C$. We also write $\operatorname{Ext}^q_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) := \operatorname{Ext}^q_{\operatorname{Tails} C}(\mathcal{M}, \mathcal{N})$ for $\mathcal{M}, \mathcal{N} \in \operatorname{Tails} C$.

If $C$ is a right coherent connected $\mathbb{Z}$-algebra, then $\operatorname{grmod} C$ is an abelian category, so $\operatorname{tors} C := \operatorname{grmod} C \cap \operatorname{Tors} C$ is a Serre subcategory of $\operatorname{grmod} C$. In this case, we define $\operatorname{tails} C := \operatorname{grmod} C/\operatorname{tors} C$.

**Lemma 3.9.** *If $C$ is a right coherent connected $\mathbb{Z}$-algebra, then* $\operatorname{tors} C$ *is the full subcategory of* $\operatorname{grmod} C$ *consisting of finite dimensional modules.*

*Proof.* Let $M \in \operatorname{tors} C$. Since $M \in \operatorname{grmod} C$, there exist $x_1, \dots, x_m \in M$ such that $M = \sum_{i=1}^m x_i C$. Since $M \in \operatorname{Tors} C$, $x_i C$ is right bounded for every $i = 0, \dots, m$. Since $C$ is locally finite by Lemma 3.5, $x_i C$ is finite dimensional for every $i = 1, \dots, m$, so $M$ is finite dimensional. The converse is clear. □

*Remark* 3.10. If $C$ is a right coherent connected $\mathbb{Z}$-algebra, then $\operatorname{tails} C$ is the same as $\operatorname{cohproj} C$ defined in [17] and [19] by Lemma 3.9, so, for the rest of the paper, we can and will view $\operatorname{tails} C$ as a full subcategory of $\operatorname{Tails} C$ by [17, Lemma 2.2 (2)].

We use the following $\mathbb{Z}$-algebra version of [4, Lemma 4.3.3] (see also [10, Proposition 1.7.11]) implicitly in the sequel.

**Lemma 3.11.** *If $C$ is a right coherent connected $\mathbb{Z}$-algebra, then the canonical functors $\mathscr{D}^b(\operatorname{grmod} C) \to \mathscr{D}(\operatorname{GrMod} C)$ and $\mathscr{D}^b(\operatorname{tails} C) \to \mathscr{D}(\operatorname{Tails} C)$ are fully faithful.*

By the above lemma, we often view $\mathscr{D}^b(\operatorname{grmod} C)$ and $\mathscr{D}^b(\operatorname{tails} C)$ as full subcategories of $\mathscr{D}(\operatorname{GrMod} C)$ and $\mathscr{D}(\operatorname{Tails} C)$, respectively.

The next result follows from a property of a localizing subcategory.

**Lemma 3.12.** (cf. [17, Theorem 6.8]) *If $C$ is a right* Ext*-finite connected $\mathbb{Z}$-algebra, then, for every $X \in \mathscr{D}^b(\operatorname{GrMod} C)$, there exists a triangle*

$$\mathrm{R}\,\tau(X) \to X \to \mathrm{R}\,Q(X)$$

*in* $\mathscr{D}(\operatorname{GrMod} C)$*. In particular, for every* $M \in \operatorname{GrMod} C$*, there exists an exact sequence*

$$0 \to \mathrm{R}^0\,\tau(M) \to M \to \mathrm{R}^0\,Q(M) \to \mathrm{R}^1\,\tau(M) \to 0,$$

*and an isomorphism* $\mathrm{R}^q\,Q(M) \cong \mathrm{R}^{q+1}\,\tau(M)$ *in* $\operatorname{GrMod} C$ *for every* $q \geq 1$*.*

We have the following analogue of [13, Lemma 6.9].

**Lemma 3.13.** *Let* $A, B$ *be* $\mathbb{Z}$*-algebras. If* $B$ *is a right Ext-finite connected* $\mathbb{Z}$*-algebra, then the left-exact functor* $Q := \omega\pi : \operatorname{GrMod} B \to \operatorname{GrMod} B$ *extends to a left-exact functor*

$$Q : \operatorname{Bimod}(A - B) \longrightarrow \operatorname{Bimod}(A - B)$$

*via functoriality of* $Q$*.*

*Proof.* Since $\mathrm{R}^i\,\tau$ commutes with direct sums by [13, Lemma 5.9], then by Lemma 3.12, $Q$ commutes with direct sums. Thus, if $M$ is an object in $\operatorname{Bimod}(A-B)$, then $Q(\oplus_i e_i M) \cong \oplus_i Q(e_i M)$ as graded right $B$-modules. Furthermore, we may define a graded left $A$-module structure on this module via functorality of $Q$, and this makes $Q(M)$ an object in $\operatorname{Bimod}(A-B)$ as one can check. It is also routine to check this defines a functor and we omit the verification. $\square$

Since $\operatorname{Bimod}(A-B)$ has enough injectives, there is a right derived functor

$$\mathrm{R}\,Q : \mathscr{D}^+(\operatorname{Bimod}(A-B)) \to \mathscr{D}(\operatorname{Bimod}(A-B)).$$

In what follows, we shall abuse notation by writing $\tau$ and $Q$ for the associated extensions of these functors to bimodule categories.

**Lemma 3.14.** *If* $C$ *is a right* Ext*-finite connected* $\mathbb{Z}$*-algebra, then there exists a triangle*

$$\mathrm{R}\,\tau(C) \to C \to \mathrm{R}\,Q(C)$$

*in* $\mathscr{D}(\operatorname{Bimod}(C-C))$*. In particular, there exist an exact sequence*

$$0 \to \mathrm{R}^0\,\tau(C) \to C \to \mathrm{R}^0\,Q(C) \to \mathrm{R}^1\,\tau(C) \to 0,$$

*and an isomorphism* $\mathrm{R}^{q+1}\,\tau(C) \cong \mathrm{R}^q\,Q(C)$ *for every* $q \geq 1$ *in* $\operatorname{Bimod}(C-C)$*.*

*Proof.* Let $I$ denote an injective resolution of $C$ in $\operatorname{Bimod}(C-C)$. For each $i \in \mathbb{Z}$, $e_i I$ is an injective resolution of $P_i$ in $\operatorname{GrMod} C$ by [13, Lemma 2.3], and thus, by Lemma 3.12, there is a short exact sequence of complexes

$$0 \to \tau(e_i I) \to e_i I \to Q(e_i I) \to 0$$

for each $i \in \mathbb{Z}$. Since $\tau$ and $Q$ commute with direct sums, then, by definition of the extensions of $\tau$ and $Q$ to bimodules, there is a short exact sequence of complexes of objects in $\operatorname{Bimod}(C-C)$

$$0 \to \tau(I) \to I \to Q(I) \to 0.$$

The result now follows from [7, Remark, p. 63]. $\square$

For a connected $\mathbb{Z}$-algebra $C$, the cohomological dimension of $\tau$ is defined by

$$\operatorname{cd}\tau := \sup\{q \in \mathbb{N} \mid \mathrm{R}^q\,\tau(M) \neq 0 \text{ for } M \in \operatorname{GrMod} C\}.$$

Note that if $\operatorname{cd}\tau < \infty$, then $D\,\mathrm{R}\,\tau(C) \in \mathscr{D}^b(\operatorname{Bimod}(C-C))$ is bounded, which is often an essential condition in the sequel.

**Lemma 3.15.** *Let $C$ be a right* Ext*-finite connected $\mathbb{Z}$-algebra. If $\operatorname{cd}\tau < \infty$, then $D\,\mathrm{R}\,Q(C) \cong \operatorname{cone}(DC \to D\,\mathrm{R}\,\tau(C))[-1]$ in $\mathscr{D}(\operatorname{Bimod}(C-C))$. In particular, $D\,\mathrm{R}\,Q(C)$ is bounded.*

*Proof.* The fact that $D\,\mathrm{R}\,\tau(C)$ is bounded follows immediately from $\operatorname{cd}\tau < \infty$. By Lemma 3.14, there is a triangle

$$\mathrm{R}\,\tau(C) \to C \to \mathrm{R}\,Q(C)$$

in $\mathscr{D}(\operatorname{Bimod}(C-C))$. Thus, there is a triangle

$$D\,\mathrm{R}\,Q(C) \to DC \to D\,\mathrm{R}\,\tau(C)$$

which may be rotated to a triangle

$$DC \to D\,\mathrm{R}\,\tau(C) \to D\,\mathrm{R}\,Q(C)[1].$$

It follows that $D\,\mathrm{R}\,Q(C) \cong \operatorname{cone}(DC \to D\,\mathrm{R}\,\tau(C))[-1]$. $\square$

*Definition* 3.16. Let $C$ be a right Ext-finite connected $\mathbb{Z}$-algebra, $\mathcal{M} \in \mathscr{D}(\operatorname{Tails} C)$ and $q \in \mathbb{Z}$.

(1) We define a graded right $C$-module structure on

$$\underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{M}) := \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(\mathcal{P}_i, \mathcal{M}[q])$$

by

$$\begin{aligned}\underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{M})_i \times C_{ij} &\cong \operatorname{Hom}_{\mathcal{C}}(\mathcal{P}_i, \mathcal{M}[q]) \times \operatorname{Hom}_C(P_j, P_i)\\ &\to \operatorname{Hom}_{\mathcal{C}}(\mathcal{P}_i, \mathcal{M}[q]) \times \operatorname{Hom}_{\mathcal{C}}(\mathcal{P}_j, \mathcal{P}_i)\\ &\to \operatorname{Hom}_{\mathcal{C}}(\mathcal{P}_j, \mathcal{M}[q])\\ &= \underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{M})_j.\end{aligned}$$

(2) We define a graded left $C$-module structure on

$$\underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{M}, \mathcal{C}) := \oplus_{j\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(\mathcal{M}[-q], \mathcal{P}_j)$$

by

$$\begin{aligned}C_{ij} \times \underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{M}, \mathcal{C})_j &\cong \operatorname{Hom}_C(P_j, P_i) \times \operatorname{Hom}_{\mathcal{C}}(\mathcal{M}[-q], \mathcal{P}_j)\\ &\to \operatorname{Hom}_{\mathcal{C}}(\mathcal{P}_j, \mathcal{P}_i) \times \operatorname{Hom}_{\mathcal{C}}(\mathcal{M}[-q], \mathcal{P}_j)\\ &\to \operatorname{Hom}_{\mathcal{C}}(\mathcal{M}[-q], \mathcal{P}_i)\\ &= \underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{M}, \mathcal{C})_i.\end{aligned}$$

**Lemma 3.17.** *If $C$ is a right* Ext*-finite connected $\mathbb{Z}$-algebra, then $\omega(-) \cong \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, -) : \operatorname{Tails} C \to \operatorname{GrMod} C$ as functors. In particular, $\mathrm{R}^q\,\omega(\mathcal{M}) \cong \underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ in $\operatorname{GrMod} C$ for every $\mathcal{M} \in \operatorname{Tails} C$ and $q \in \mathbb{Z}$.*

*Proof.* For $\mathcal{M} \in \operatorname{Tails} C$,

$$\begin{aligned}\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{M}) &:= \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}}(\mathcal{P}_i, \mathcal{M}) \cong \oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{C}(P_i, \omega(\mathcal{M})) \\ &=: \underline{\operatorname{Hom}}_{C}(C, \omega(\mathcal{M})) \cong \omega(\mathcal{M})\end{aligned}$$

as graded vector spaces where the last isomorphism is in $\operatorname{GrMod} C$ by Lemma 2.2 (1). Since $\pi : \operatorname{GrMod} C \to \operatorname{Tails} C$ is a functor such that $\pi\omega\mathcal{M} \cong \mathcal{M}$, we have the following commutative diagram

$$\begin{tikzcd}
(\omega\mathcal{M})_i \times C_{ij} \arrow[r] \arrow[d, equal] & (\omega\mathcal{M})_j \arrow[d, equal] \\
\operatorname{Hom}_C(P_i, \omega\mathcal{M}) \times \operatorname{Hom}_C(P_j, P_i) \arrow[r] \arrow[d] & \operatorname{Hom}_C(P_j, \omega\mathcal{M}) \arrow[d] \\
\operatorname{Hom}_{\mathcal{C}}(\mathcal{P}_i, \mathcal{M}) \times \operatorname{Hom}_{\mathcal{C}}(\mathcal{P}_j, \mathcal{P}_i) \arrow[r] & \operatorname{Hom}_{\mathcal{C}}(\mathcal{P}_j, \mathcal{M}) \arrow[d, equal] \\
\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})_i \times C_{ij} \arrow[r] \arrow[u] & \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})_j
\end{tikzcd}$$

so $\omega\mathcal{M} \cong \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ in $\operatorname{GrMod} C$. It follows that

$$\omega(-) \cong \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, -) : \operatorname{Tails} C \to \operatorname{GrMod} C$$

as functors, so $\mathrm{R}\,\omega\mathcal{M} \cong \mathrm{R}\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ in $\mathscr{D}(\operatorname{GrMod} C)$, hence $\mathrm{R}^q\,\omega\mathcal{M} = h^q(\mathrm{R}\,\omega\mathcal{M}) \cong h^q(\mathrm{R}\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{M})) = \underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{M})$ in $\operatorname{GrMod} C$ for every $q \in \mathbb{Z}$. $\square$

*Remark* 3.18. If $C$ is a right Ext-finite connected $\mathbb{Z}$-algebra, then $\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$ has a structure of a bigraded $C$-bimodule as defined in Definition 3.16. On the other hand, since $\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \pi(-)) : \operatorname{GrMod} C \to \operatorname{GrMod} C$ is a functor commuting with direct sums, $\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$ has a structure of a bigraded $C$-bimodule as defined in the proof of Lemma 3.13. It is routine to check that these bigraded $C$-bimodule structures are the same.

The following result plays a key role in the proof of Proposition 4.11, which gives necessary and sufficient conditions for an algebra to satisfy various regularity conditions.

**Lemma 3.19.** *If $C$ is a right* Ext*-finite connected $\mathbb{Z}$-algebra, then*

$$\mathrm{R}\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \cong \mathrm{R}\,Q(C).$$

*Therefore, there exists a triangle*

$$\mathrm{R}\,\tau(C) \to C \to \mathrm{R}\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$$

*in $\mathscr{D}(\operatorname{Bimod}(C - C))$. In particular, there exists an exact sequence*

$$0 \to \mathrm{R}^0\,\tau(C) \to C \to \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \to \mathrm{R}^1\,\tau(C) \to 0$$

*and an isomorphism $\mathrm{R}^{q+1}\,\tau(C) \cong \underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$ for every $q \geq 1$ in $\operatorname{Bimod}(C - C)$.*

*Proof.* The second and third part of the result will follow from the first by Lemma 3.14. To prove the first part of the result, we know that $Q(-) \cong \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \pi(-)) : \mathrm{GrMod}\, C \to \mathrm{GrMod}\, C$ as functors by Lemma 3.17, so that, if we extend $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \pi(-))$ to an endofunctor on $\mathrm{Bimod}(C - C)$ using functoriality as we did in the proof of Lemma 3.13, we have $Q(-) \cong \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \pi(-)) : \mathrm{Bimod}(C - C) \to \mathrm{Bimod}(C - C)$ as functors. □

3.2. **Local Duality.** We will need the following notation from [13]: viewing $k$ as a graded algebra concentrated in degree 0, we define a connected $\mathbb{Z}$-algebra $K := \overline{k}$, that is,

$$K_{ij} = k_{j-i} = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Note that $K^o = K$ as $\mathbb{Z}$-algebras, so $\mathrm{Bimod}(K^o - C) = \mathrm{Bimod}(K - C)$ denotes the category of bigraded $K$-$C$-bimodules.

In [13] and [14], local duality is proven for objects of $\mathscr{D}^-(\mathrm{Bimod}(K - C))$. However, for various applications in this paper, we need to apply it to objects of $\mathscr{D}^-(\mathrm{GrMod}\, C)$. For this reason we introduce the functor

$$I_i(-) := Ke_i \otimes_k - : \mathrm{GrMod}\, C \to \mathrm{Bimod}(K - C).$$

**Lemma 3.20.** *Let $C$ be a $\mathbb{Z}$-algebra.*

(1) $I_i : \mathrm{GrMod}\, C \to \mathrm{Bimod}(K - C)$ *and* $e_j(-) : \mathrm{Bimod}(K - C) \to \mathrm{GrMod}\, C$ *are exact functors, which induce functors* $I_i : \mathscr{D}(\mathrm{GrMod}\, C) \to \mathscr{D}(\mathrm{Bimod}(K-C))$ *and* $e_j(-) : \mathscr{D}(\mathrm{Bimod}(K-C)) \to \mathscr{D}(\mathrm{GrMod}\, C)$ *such that*
$$e_j(-) \circ I_i \cong \begin{cases} \mathrm{id} & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$
(2) *For every $i, j \in \mathbb{Z}$, $I_i(P_j) \in \mathrm{Bimod}(K - C)$ is a projective bigraded $K$-$C$ bimodule.*
(3) *$I_i$ induces a fully faithful functor $\mathscr{D}(\mathrm{GrMod}\, C) \to \mathscr{D}(\mathrm{Bimod}(K - C))$.*

*Proof.* (1) Clearly, $I_i$ and $e_j(-)$ are exact functors. If $M \in \mathrm{GrMod}\, C$, then

$$\begin{aligned} e_j I_i(M) &= \oplus_{s \in \mathbb{Z}} (Ke_i \otimes_k M)_{js} \\ &= \oplus_{s \in \mathbb{Z}} ((Ke_i)_j \otimes_k M_s) \\ &= \begin{cases} \oplus_{s \in \mathbb{Z}} M_s = M & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

(2) By [13, Lemma 2.4], $I_i(P_j) = Ke_i \otimes_k e_j C$ is a projective bigraded $K$-$C$ bimodule.

Part (3) follows immediately from (1). □

Using the lemma, we have the following consequence of [14, Theorem 2.1].

**Theorem 3.21.** *Let $C$ be a right* Ext*-finite connected $\mathbb{Z}$-algebra such that* $\mathrm{cd}\, \tau < \infty$. *For $M \in \mathscr{D}^-(\mathrm{GrMod}\, C)$,*

$$D\, \mathrm{R}\, \tau(M) \cong \mathrm{R}\underline{\mathrm{Hom}}_C(M, D\, \mathrm{R}\, \tau(C))$$

*in* $\mathscr{D}(\operatorname{GrMod} C^o)$.

*Proof.* For $M \in \mathscr{D}^-(\operatorname{GrMod} C)$,

$$\begin{aligned} D\,\mathrm{R}\,\tau(M) &\cong D\,\mathrm{R}\,\tau(e_0 I_0(M)) && \text{[Lemma 3.20 (1)]} \\ &\cong D(e_0\,\mathrm{R}\,\tau(I_0(M))) \\ &\cong (D\,\mathrm{R}\,\tau(I_0(M))e_0 \\ &\cong \mathrm{R}\underline{\mathrm{Hom}}_C(I_0(M), D\,\mathrm{R}\,\tau(C))e_0 && \text{[14, Theorem 2.1]} \\ &\cong \mathrm{R}\underline{\mathrm{Hom}}_C(e_0 I_0(M), D\,\mathrm{R}\,\tau(C)) \\ &\cong \mathrm{R}\underline{\mathrm{Hom}}_C(M, D\,\mathrm{R}\,\tau(C)) && \text{[Lemma 3.20 (1)]} \end{aligned}$$

in $\mathscr{D}(\operatorname{GrMod} C^o)$. □

For a connected $\mathbb{Z}$-algebra $C$, we define the small global dimension of $C$ by

$$\operatorname{sgldim} C := \sup\{\operatorname{pd}(S_i) \mid i \in \mathbb{Z}\}.$$

For the readers convenience, we recall [14, Lemma 2.2]:

**Lemma 3.22.** *Let $C$ be a right Ext-finite connected $\mathbb{Z}$-algebra. If* $\operatorname{sgldim} C < \infty$*, then* $\operatorname{cd} \tau < \infty$.

**Lemma 3.23.** *Let $C$ be a connected $\mathbb{Z}$-algebra. If* $\operatorname{sgldim} C < \infty$*, then the global dimensions of* $\operatorname{GrMod} C$ *and* $\operatorname{GrMod} C^o$ *are finite.*

*Proof.* By [13, Corollary 4.10], $\operatorname{sgldim} C^o = \operatorname{sgldim} C < \infty$. Since the functor $-\underline{\otimes}_C C/C_{\geq 1} : \operatorname{GrMod} C \longrightarrow \operatorname{GrMod} C$ commutes with direct limits, the left derived functors $\underline{\mathrm{Tor}}_i^C(-, C/C_{\geq 1})$, $i > 0$ defined in [13, Section 4.2], also commute with direct limits by the $\mathbb{Z}$-algebra analogue of the usual argument in the graded context. Therefore, since every graded module is a direct limit of left-bounded modules, the fact that the global dimension of $\operatorname{GrMod} C$ is finite follows by [13, Proposition 4.11], [13, Proposition 4.7], and the fact that $\operatorname{sgldim} C^o < \infty$. The fact that the global dimension of $\operatorname{GrMod} C^o$ is finite now follows by symmetry. □

**Lemma 3.24.** *Let $C$ be a coherent connected $\mathbb{Z}$-algebra. If* $\operatorname{sgldim} C < \infty$*, then we have a duality*

$$\mathrm{R}\underline{\mathrm{Hom}}_C(-, C) : \mathscr{D}^b(\operatorname{grmod} C) \leftrightarrow \mathscr{D}^b(\operatorname{grmod} C^o) : \mathrm{R}\underline{\mathrm{Hom}}_{C^o}(-, C^o).$$

*Proof.* By [13, Corollary 4.10], $\operatorname{sgldim} C^o = \operatorname{sgldim} C < \infty$. Thus, by Lemma 3.23 and [7, p. 68, Example 1],

$$\mathrm{R}\underline{\mathrm{Hom}}_{C^o}(\mathrm{R}\,\underline{\mathrm{Hom}}_C(-, C), C^o) : \mathscr{D}(\operatorname{GrMod} C) \to \mathscr{D}(\operatorname{GrMod} C)$$

is way out on both sides. Furthermore, $\underline{\mathrm{Hom}}_C(-, C) : \operatorname{GrMod} C \to \operatorname{GrMod} C^o$ equals the functor $\underline{\mathrm{Hom}}_C(I_0(-), C)e_0$ when $\underline{\mathrm{Hom}}_C(-, C)$ is considered as a functor from $\operatorname{Bimod}(K - C)$ to $\operatorname{Bimod}(C - K)$. Thus, as in the proof of [14, Theorem 3.7], the functors $\mathrm{R}\underline{\mathrm{Hom}}_C(-, C)$ and $\mathrm{R}\underline{\mathrm{Hom}}_{C^o}(-, C^o)$ induce functors between the categories $\mathscr{D}^b(\operatorname{grmod} C)$ and $\mathscr{D}^b(\operatorname{grmod} C^o)$ by Lemma 3.11. The result now follows from [14, Lemma 3.6 (2)] in the case $l = 0$ and $\nu = \mathrm{id}$ (see the argument in the proof of [14, Theorem 3.7]). □

*Definition* 3.25. We say $L \in \mathscr{D}^b(\mathrm{Bimod}(K - C))$ is a perfect complex if the terms of $L$ are finite direct sums of modules of the form $Ke_i \otimes_k e_jC$.

**Lemma 3.26.** *Let $C$ be a right coherent connected $\mathbb{Z}$-algebra. If* $\operatorname{sgldim} C < \infty$*, then every $X \in \mathscr{D}^b(\operatorname{grmod} C)$ has a finitely generated free resolution of finite length. Therefore, $I_0(X)$ is quasi-isomorphic to a perfect complex.*

*Proof.* For $X \in \mathscr{D}^b(\operatorname{grmod} C)$, we may assume that $X^q = 0$ for all $q \gg 0$ and all $q \ll 0$. Since $X^q \in \operatorname{grmod} C$ is left bounded, $\operatorname{pd}(X^q) < \infty$ by [13, Proposition 4.11], so $X^q$ has a unique minimal finitely generated free resolution of finite length for every $q \in \mathbb{Z}$ by Lemma 3.2 and [13, Corollary 4.5]. The total complex of the Cartan-Eilenberg resolution of $X$ is a finitely generated free resolution of $X$ of finite length. $\square$

The following result, which will be employed to prove Theorem 6.6, is a $\mathbb{Z}$-algebra version of [9, Proposition 2.1]. The proof employs the functors $\underline{\mathrm{Hom}}^{\bullet}_C(-,-)$ and $\mathrm{Tot}(-\underline{\otimes}_C-)$ defined in [13, Section 6].

**Proposition 3.27.** *Let $B$ and $C$ be $\mathbb{Z}$-algebras, $X \in \mathscr{D}^b(\mathrm{Bimod}(K - C))$, $Y \in \mathscr{D}^b(\mathrm{Bimod}(B{-}C))$ and $Z \in \mathscr{D}^-(\mathrm{Bimod}(K{-}B))$. If $X$ is quasi-isomorphic to a perfect complex, then there is an isomorphism*

$$\mathrm{R}\,\underline{\mathrm{Hom}}_C(X, Z\underline{\otimes}^{\mathrm{L}}_B Y) \cong Z\underline{\otimes}^{\mathrm{L}}_B \mathrm{R}\,\underline{\mathrm{Hom}}_C(X, Y).$$

*Proof.* By assumption, $X$ is quasi-isomorphic to a perfect complex $L$. Moreover, by [13, Lemma 2.4], $Z$ is quasi-isomorphic to a bounded above complex $F$, whose terms are direct sums of modules of the form $Ke_i \otimes_k e_jB$. Thus, by [13, Proposition 6.6 (2), Proposition 6.8],

$$\mathrm{R}\,\underline{\mathrm{Hom}}_C(X, Z\underline{\otimes}^{\mathrm{L}}_B Y) \cong \underline{\mathrm{Hom}}^{\bullet}_C(L, \mathrm{Tot}(F\underline{\otimes}_B Y)),$$

and

$$Z\underline{\otimes}^{\mathrm{L}}_B \mathrm{R}\,\underline{\mathrm{Hom}}_C(X, Y) \cong \mathrm{Tot}(F\underline{\otimes}_B \underline{\mathrm{Hom}}^{\bullet}_C(L, Y)).$$

Therefore, it suffices to prove that there is a natural isomorphism of complexes

$$\underline{\mathrm{Hom}}^{\bullet}_C(L, \mathrm{Tot}(F\underline{\otimes}_B Y)) \cong \mathrm{Tot}(F\underline{\otimes}_B \underline{\mathrm{Hom}}^{\bullet}_C(L, Y)).$$

Since, for $P$ a free module in $\mathrm{GrMod}B$, $M$ in $\mathrm{GrMod}C$ a finitely generated free module, and $N$ in $\mathrm{Bimod}(B - C)$, there is a natural isomorphism

$$P\underline{\otimes}_B \underline{\mathrm{Hom}}_C(M, N) \longrightarrow \underline{\mathrm{Hom}}_C(M, P\underline{\otimes}_B N),$$

in $\mathrm{GrMod}C$, the remainder of the proof is the same as that of [9, Proposition 2.1]. $\square$

We also will employ the following variant of [13, Lemma 6.10].

**Lemma 3.28.** *Let $C$ be a right Ext-finite connected $\mathbb{Z}$-algebra. If $M$ is a complex of free right $C$-modules, and $N$ is a complex of bigraded $C$-bimodules, then there exists a canonical isomorphism*

$$\mathrm{Tot}(M\underline{\otimes}_C Q(N)) \cong Q(\mathrm{Tot}(M\underline{\otimes}_C N)).$$

*Proof.* By Lemma 3.12 and [13, Lemma 5.9], $\mathrm{R}^i Q$ commutes with direct limits for $i \geq 0$. Thus, the argument in [13, Lemma 6.10] can be applied to prove the result. □

The following is a $\mathbb{Z}$-algebra version of local duality found in [16].

**Theorem 3.29.** *Let $C$ be a right* Ext*-finite connected $\mathbb{Z}$-algebra such that* $\operatorname{cd}\tau < \infty$. *For $M \in \mathscr{D}^-(\mathrm{Bimod}(K - C))$,*

$$D\,\mathrm{R}\,Q(M) \cong \mathrm{R}\underline{\mathrm{Hom}}_C(M, D\,\mathrm{R}\,Q(C))$$

*in $\mathscr{D}(\mathrm{Bimod}(C - K))$.*

*Proof.* Since $\operatorname{cd}\tau < \infty$, the cohomological dimension of $Q$ is finite by Lemma 3.12. Furthermore, by Lemma 3.12 and [13, Lemma 5.9], $\mathrm{R}^i Q$ commutes with direct limits for $i \geq 0$. Thus, by Lemma 3.28, the argument of [14, Theorem 2.1] can be applied to prove the result. □

## 4. AS-regular $\mathbb{Z}$-algebras

AS-regular algebras were originally introduced in [1], and play an essential role in noncommutative algebraic geometry. The related notion of ASF-regular algebras was originally introduced in [11]. After recalling these notions, we modify these definitions for the purpose of this paper.

*Definition* 4.1. Let $A$ be a locally finite connected graded algebra.

(1) $A$ is called *AS-regular of dimension $d$ and of Gorenstein parameter $\ell$* if
   - $\operatorname{gldim} A = d$, and
   - $\mathrm{Ext}^q_A(k, A(j)) \cong \begin{cases} k & \text{if } q = d \text{ and } j = -\ell, \\ 0 & \text{otherwise.} \end{cases}$

(2) $A$ is called *ASF-regular of dimension $d$ and of Gorenstein parameter $\ell$* if
   - $\operatorname{gldim} A = d$, and
   - $D\,\mathrm{R}^q\,\tau(A) \cong \begin{cases} A(-\ell) & \text{if } q = d, \\ 0 & \text{otherwise} \end{cases}$ as graded left and right $A$-modules.

Recall that a connected graded algebra $A$ is right Ext-finite if and only if $A$ is left Ext-finite by [25], so we may just say that $A$ is Ext-finite, and, in this case, $A$ is locally finite. If $A$ is an Ext-finite connected graded algebra, then $A$ is AS-regular of dimension $d$ and of Gorenstein parameter $\ell$ if and only if $A$ is ASF-regular of dimension $d$ and of Gorenstein parameter $\ell$ by [11, Theorem 3.12] and [15, Theorem 2.19].

In this section, we extend these definitions to $\mathbb{Z}$-algebras, and study some properties and relationships of these $\mathbb{Z}$-algebras.

4.1. **AS-regular $\mathbb{Z}$-algebras and ASF-regular $\mathbb{Z}$-algebras.** The following definition was given in [13].

*Definition* 4.2. [13, Definition 7.1] A locally finite connected $\mathbb{Z}$-algebra $C$ is called *AS-regular of dimension $d$ and of Gorenstein parameter $\ell$* if

(ASR1) $\operatorname{pd}_C S_i = d$ for every $i \in \mathbb{Z}$, and

(ASR2) $\operatorname{Ext}^q_C(S_i, P_j) = \begin{cases} k & \text{if } q = d \text{ and } j = i+\ell \\ 0 & \text{otherwise,} \end{cases}$ that is,

$$\operatorname{RHom}_C(S_i, P_j) \cong \begin{cases} k[-d] & \text{if } j = i+\ell \\ 0 & \text{otherwise} \end{cases}$$

in $\mathscr{D}(\operatorname{Mod} k)$.

*Remark* 4.3. Since $\underline{\operatorname{Ext}}^q_C(S_i, C)_j = \operatorname{Ext}^q_C(S_i, e_jC) = \operatorname{Ext}^q_C(S_i, P_j)$, the condition (ASR2) is equivalent to

$$\underline{\operatorname{Ext}}^q_C(S_i, C) \cong \begin{cases} S_{i+\ell} & \text{if } q = d \\ 0 & \text{otherwise} \end{cases}$$

as graded left $C$-modules for every $i \in \mathbb{Z}$.

The following proposition justifies our definition of an AS-regular $\mathbb{Z}$-algebra above. A similar result was stated in [26, Section 4.1] although the definition of an AS-regular $\mathbb{Z}$-algebra in [26, Section 4.1] differs slightly from ours. (In [26], the Gorenstein parameter is not well-defined.)

**Proposition 4.4.** *Let $A$ be a locally finite connected graded algebra. Then $A$ is an AS-regular algebra of dimension $d$ and of Gorenstein parameter $\ell$ if and only if $\overline{A}$ is an AS-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$.*

It follows from the above proposition and Lemma 2.20 that $C$ is a 1-periodic AS-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$ if and only if there exists an AS-regular algebra $A$ of dimension $d$ and of Gorenstein parameter $\ell$ such that $C \cong \overline{A}$.

The next result tells us that AS-regularlity is left-right symmetric.

**Theorem 4.5.** *Let $C$ be a connected $\mathbb{Z}$-algebra. Then $C$ is a right* Ext*-finite AS-regular algebra of dimension $d$ and of Gorenstein parameter $\ell$ if and only if $C^o$ is a right* Ext*-finite AS-regular algebra of dimension $d$ and of Gorenstein parameter $\ell$.*

*Proof.* Let $C$ be a right Ext-finite AS-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$. Since $C$ is right Ext-finite, $S_i$ has a minimal finitely generated free resolution

$$0 \to F^d \to \cdots \to F^0 \to S_i \to 0$$

in $\operatorname{GrMod} C$. Since $F^q \in \operatorname{add}\{P_j\}_{j\in\mathbb{Z}}$, we have $\underline{\operatorname{Hom}}_C(F^q, C) \in \operatorname{add}\{Q_j\}_{j\in\mathbb{Z}} = \operatorname{add}\{P^o_j\}_{j\in\mathbb{Z}}$ for every $q$ by Lemma 2.2 (2). By Remark 4.3,

$$0 \to \underline{\operatorname{Hom}}_C(F^0, C) \to \cdots \to \underline{\operatorname{Hom}}_C(F^d, C) \to \underline{\operatorname{Ext}}^d_C(S_i, C) \cong S_{i+\ell} \to 0$$

is a minimal finitely generated free resolution of $S^o_{-\ell-i} \cong S_{i+\ell}$ in $\operatorname{GrMod} C^o$, so $C^o$ is right Ext-finite and $\operatorname{pd} S^o_{-\ell-i} = d$ for every $i \in \mathbb{Z}$.

Since $\mathrm{R}\,\underline{\operatorname{Hom}}_{C^o}(\mathrm{R}\,\underline{\operatorname{Hom}}_C(F, C), C^o) \cong F$ by [14, Lemma 3.6 (2)] (see Lemma 3.24),

$$0 \to \underline{\operatorname{Hom}}_{C^o}(\underline{\operatorname{Hom}}_C(F^d, C), C^o) \to \cdots \to \underline{\operatorname{Hom}}_{C^o}(\underline{\operatorname{Hom}}_C(F^0, C), C^o) \to S_i \to 0$$

is isomorphic to the original minimal finitely generated free resolution of $S_i$ in $\operatorname{GrMod} C$ by the uniqueness of the minimal free resolution, so

$$\underline{\operatorname{Ext}}^q_{C^o}(S^o_{-\ell-i}, C^o) = \begin{cases} S_i \cong S^o_{-i} & \text{if } q = d \\ 0 & \text{otherwise,} \end{cases}$$

hence $C^o$ is an AS-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$ by Remark 4.3. □

*Remark* 4.6. In the setting of the above theorem, since

$$\operatorname{Hom}_C(P_i, S_j) = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

by Lemma 2.2 (1), we have $F^0 \cong P_i$. Similarly, since

$$\operatorname{Hom}_{C^o}(P^o_i, S^o_j) = \begin{cases} k & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

we have $\underline{\operatorname{Hom}}_C(F^d, C) \cong P^o_{-\ell-i} \cong Q_{i+\ell}$, so we have $F^d \cong P_{i+\ell}$. It follows that the minimal finitely generated free resolution of $S_i$ in $\operatorname{GrMod} C$ is of the form

$$0 \to P_{i+\ell} \to \cdots \to P_i \to S_i \to 0.$$

The following definition was also given in [13].

*Definition* 4.7. [13, Definition 7.5] A connected $\mathbb{Z}$-algebra $C$ is called *ASF-regular of dimension $d$ and of Gorenstein parameter $\ell$* if

(ASF1) $\operatorname{sgldim} C = d < \infty$, and

(ASF2) $\mathrm{R}^q\,\tau(P_j) \cong \begin{cases} D(Q_{j-\ell}) & \text{if } q = d \\ 0 & \text{otherwise} \end{cases}$ as graded right $C$-modules for every $j \in \mathbb{Z}$, that is,

$$\mathrm{R}\,\tau(P_j) \cong (D(Q_{j-\ell}))[-d]$$

in $\mathscr{D}(\operatorname{GrMod} C)$.

As to relationships between AS-regular $\mathbb{Z}$-algebras and ASF-regular $\mathbb{Z}$-algebras, one implication was proved in [13].

**Theorem 4.8.** [13, Corollary 7.7] *If $C$ is a right* Ext*-finite AS-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$, then $C$ is an ASF-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$.*

4.2. **ASF$^+$-regular $\mathbb{Z}$-algebras.** We now modify the original definition of an ASF-regular algebra given in Definition 4.1, replacing the condition (ASF2) with conditions that takes into account both the left and right $C$-module structure on $\mathrm{R}\,\tau(C)$.

The following definition is closer to the original definition of an ASF-regular algebra.

*Definition* 4.9. A locally finite connected $\mathbb{Z}$-algebra $C$ is called *ASF$^+$-regular of dimension $d$ and of Gorenstein parameter $\ell$* if

(ASF1) $\operatorname{sgldim} C = d < \infty$,

(ASF2) $(D\,\mathrm{R}\,\tau(C))e_j \cong Ce_{j-\ell}[d] = Q_{j-\ell}[d]$ in $\mathscr{D}(\operatorname{GrMod} C^o)$ for every $j \in \mathbb{Z}$, and

(ASF2$^+$) $e_i(D\,\mathrm{R}\,\tau(C)) \cong e_{i+\ell}C[d] = P_{i+\ell}[d]$ in $\mathscr{D}(\operatorname{GrMod} C)$ for every $i \in \mathbb{Z}$.

*Remark* 4.10. Let $C$ be a connected $\mathbb{Z}$-algebra.

(1) Since
$$(D\,\mathrm{R}\,\tau(C))e_j \cong D(e_j\,\mathrm{R}\,\tau(C)) \cong D(\mathrm{R}\,\tau(e_jC)) = D\,\mathrm{R}\,\tau(P_j)$$
as graded left $C$-modules, the above condition (ASF2) is equivalent to the condition (ASF2) in Defintion 4.7. However, since $Q_j$ has no graded right $C$-module structure, $\tau(Q_j)$ is not well-defined, so we are not able to replace $e_i(D\,\mathrm{R}\,\tau(C))$ by $D\,\mathrm{R}\,\tau(Q_i)$ in the above condition (ASF2$^+$).

(2) Since $C(0,-\ell)e_j = \oplus_{i\in\mathbb{Z}} C_{i,j-\ell} = Q_{j-\ell}$ as graded left $C$-modules and $e_iC(\ell,0) = \oplus_{j\in\mathbb{Z}} C_{i+\ell,j} = P_{i+\ell}$ as graded right $C$-modules, the condition (ASF2) is equivalent to
$$D\,\mathrm{R}^q\,\tau(C) \cong \begin{cases} C(0,-\ell) & \text{if } q = d \\ 0 & \text{if } q \neq d \end{cases}$$
as graded left $C$-modules, and the condition (ASF2$^+$) is equivalent to
$$D\,\mathrm{R}^q\,\tau(C) \cong \begin{cases} C(\ell,0) & \text{if } q = d \\ 0 & \text{if } q \neq d \end{cases}$$
as graded right $C$-modules.

(3) Recall that $C(0,-\ell)$ has a bigraded $C$-$C(-\ell)$ bimodule structure, and $C(\ell,0)$ has a bigraded $C(\ell)$-$C$ bimodule structure. Although $D\,\mathrm{R}^d\,\tau(C)$ has a bigraded $C$-bimodule structure, $C(0,-\ell), C(\ell,0)$ do not have natural bigraded $C$-bimodule structures unless $C$ is $\ell$-periodic. Even if $C$ is $\ell$-periodic, $C(0,-\ell)$ and $C(\ell,0)$ are not isomorphic to $C$ as bigraded $C$-bimodules in general.

**Proposition 4.11.** *Let $C$ be a right* Ext*-finite connected $\mathbb{Z}$-algebra of* $\operatorname{sgldim} C = d$.

(1) *For* $d = 1$,
- $C$ *satisfies (ASF2) if and only if there is an exact sequence*
$$0 \to C \to \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \to D(C(0, -\ell)) \to 0$$
*as graded right* $C$*-modules and* $\underline{\mathrm{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) = 0$ *for every* $q \geq 1$.
- $C$ *satisfies (ASF2*$^+$*) if and only if there is an exact sequence*
$$0 \to C \to \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \to D(C(\ell, 0)) \to 0$$
*as graded left* $C$*-modules and* $\underline{\mathrm{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) = 0$ *for every* $q \geq 1$.

(2) *For* $d \geq 2$,
- $C$ *satisfies (ASF2) if and only if*
$$\underline{\mathrm{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \cong \begin{cases} C & \text{if } q = 0 \\ D(C(0, -\ell)) & \text{if } q = d-1 \\ 0 & \text{otherwise} \end{cases}$$
*as graded right* $C$*-modules, and*
- $C$ *satisfies (ASF2*$^+$*) if and only if*
$$\underline{\mathrm{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \cong \begin{cases} C & \text{if } q = 0 \\ D(C(\ell, 0)) & \text{if } q = d-1 \\ 0 & \text{otherwise} \end{cases}$$
*as graded left* $C$*-modules.*

*In the above, we tacitely require that the isomorphism when* $q = 0$ *is induced by the canonical map* $C \cong \underline{\mathrm{Hom}}_C(C, C) \to \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$.

*Proof.* By Lemma 3.19 and Lemma 4.10 (2), we have a triangle
$$(D(C(0, -\ell)))[-d] \cong \mathrm{R}\,\tau(C) \to C \to \mathrm{R}\,\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$$
in $\mathscr{D}(\mathrm{GrMod}\, C)$ and a triangle
$$(D(C(\ell, 0)))[-d] \cong \mathrm{R}\,\tau(C) \to C \to \mathrm{R}\,\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$$
in $\mathscr{D}(\mathrm{GrMod}\, C^o)$, so the result follows. □

**Lemma 4.12.** *Let* $A$ *be an* Ext*-finite connected graded algebra, and* $\Phi : \mathrm{GrMod}\, A \to \mathrm{GrMod}\, \overline{A}, \Phi^o : \mathrm{GrMod}\, A^o \to \mathrm{GrMod}\, \overline{A}^o$ *equivalence functors defined in Lemma 2.17 and Lemma 2.18.*

(1) $\overline{A}$ *is right* Ext*-finite.*
(2) $\Phi\tau \cong \tau\Phi : \mathrm{GrMod}\, A \to \mathrm{GrMod}\, \overline{A}$ *as functors.*
(3) *For* $M \in \mathrm{GrMod}\, A^e$, $e_i \mathrm{R}\,\tau(\overline{M}) \cong \Phi(\mathrm{R}\,\tau(M(-i)))$ *for every* $i \in \mathbb{Z}$ *in* $\mathscr{D}(\mathrm{GrMod}\, \overline{A})$, *and* $\mathrm{R}\,\tau(\overline{M})e_j \cong \Phi^o(\mathrm{R}\,\tau(M(j)))$ *for every* $j \in \mathbb{Z}$ *in* $\mathscr{D}(\mathrm{GrMod}\, \overline{A}^o)$.

*Proof.* (1) Since $\Phi(A(-i)) \cong P_i$, if $F$ is a minimal finitely generated free resoution of $k(j)$ in $\mathrm{GrMod}\, A$, then $\Phi(F)$ is a minimal finitely generated free resoution of $\Phi(k(-i)) \cong S_i$ in $\mathrm{GrMod}\, \overline{A}$ by Lemma 2.17.

(2) Since $M$ is right bounded if and only if $\Phi(M)$ is right bounded, $\Phi$ restricts to an equivalence functor $\Phi : \mathrm{Tors}\, A \to \mathrm{Tors}\, \overline{A}$. Since $\tau : \mathrm{GrMod}\, A \to$

$\operatorname{Tors} A$, and $\tau : \operatorname{GrMod} \overline{A} \to \operatorname{Tors} \overline{A}$ are right adjoint to the inclusion functors $\operatorname{Tors} A \to \operatorname{GrMod} A$ and $\operatorname{Tors} \overline{A} \to \operatorname{GrMod} \overline{A}$, respectively, we have $\Phi\tau \cong \tau\Phi : \operatorname{GrMod} A \to \operatorname{GrMod} \overline{A}$ as functors.

(3) For $M \in \operatorname{GrMod} A^e$,

$$e_i\tau(\overline{M}) \cong \tau(e_i\overline{M}) \cong \tau\Phi(M(-i)) \cong \Phi\tau(M(-i))$$

in $\operatorname{GrMod} \overline{A}$ by Lemma 2.18 (2), and (2) above, so

$$e_i(-) \circ \tau \circ \overline{(-)} \cong \Phi \circ \tau \circ (-i) : \operatorname{GrMod} A^e \to \operatorname{GrMod} \overline{A}$$

as functors for every $i \in \mathbb{Z}$. Since functors $\overline{(-)} : \operatorname{GrMod} A^e \to \operatorname{Bimod}(\overline{A} - \overline{A})$, $e_i(-) : \operatorname{GrMod} \overline{A}^e \to \operatorname{GrMod} \overline{A}$, $(-i) : \operatorname{GrMod} A^e \to \operatorname{GrMod} A^e$, and $\Phi : \operatorname{GrMod} A^e \to \operatorname{GrMod} \overline{A}$ are all exact, $e_i \mathrm{R}\,\tau(\overline{M}) \cong \Phi(\mathrm{R}\,\tau(M(-i)))$ for every $i \in \mathbb{Z}$ in $\mathscr{D}(\operatorname{GrMod} \overline{A})$.

On the other hand, since $\overline{(-)} : \operatorname{GrMod} A \to \operatorname{GrMod} \overline{A}$ is an exact functor, and

$$(\overline{A_{\geq n}})_{ij} = (A_{\geq n})_{j-i} = \begin{Bmatrix} A_{j-i} & \text{if } j-i \geq n \\ 0 & \text{if } j-i < n \end{Bmatrix} = \begin{Bmatrix} \overline{A}_{ij} & \text{if } j-i \geq n \\ 0 & \text{if } j-i < n \end{Bmatrix} = (\overline{A}_{\geq n})_{ij}$$

for every $n \in \mathbb{Z}$,

$$\overline{A/A_{\geq n}} \cong \overline{A}/\overline{A_{\geq n}} \cong \overline{A}/\overline{A}_{\geq n},$$

so

$$\begin{aligned} \tau(\overline{M})e_j &\cong \lim_{n\to\infty} \underline{\operatorname{Hom}}_{\overline{A}}(\overline{A}/\overline{A}_{\geq n}, \overline{M})e_j \\ &= \lim_{n\to\infty} \underline{\operatorname{Hom}}_{\overline{A}}(e_j(\overline{A}/\overline{A}_{\geq n}), \overline{M}) \\ &:= \lim_{n\to\infty} (\oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\overline{A}}(e_j\overline{A/A_{\geq n}}, e_i\overline{M})) \\ &\cong \lim_{n\to\infty} (\oplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\overline{A}}(\Phi((A/A_{\geq n})(-j)), \Phi(M(-i))) \\ &\cong \lim_{n\to\infty} \Phi^o(\oplus_{i\in\mathbb{Z}} \operatorname{Hom}_A((A/A_{\geq n})(-j), M(i))) \\ &= \Phi^o(\lim_{n\to\infty} \underline{\operatorname{Hom}}_A(A/A_{\geq n}, M(j))) \\ &\cong \Phi^o(\tau(M(j))) \end{aligned}$$

in $\operatorname{GrMod} \overline{A}^o$ by [13, Lemma 5.8], Lemma 2.18 (2), and Lemma 2.17, hence

$$(-)e_j \circ \tau \circ \overline{(-)} \cong \Phi^o \circ \tau \circ (j) : \operatorname{GrMod} A^e \to \operatorname{GrMod} \overline{A}^o$$

as functors for every $j \in \mathbb{Z}$.

Since the functors $\overline{(-)} : \operatorname{GrMod} A^e \to \operatorname{Bimod}(\overline{A} - \overline{A})$, $(-)e_j : \operatorname{GrMod} \overline{A}^e \to \operatorname{GrMod} \overline{A}^o$, $(j) : \operatorname{GrMod} A^e \to \operatorname{GrMod} A^e$, and $\Phi^o : \operatorname{GrMod} A^e \to \operatorname{GrMod} \overline{A}^o$ are all exact, $\mathrm{R}\,\tau(\overline{M})e_j \cong \Phi^o(\mathrm{R}\,\tau(M(j)))$ for every $i \in \mathbb{Z}$ in $\mathscr{D}(\operatorname{GrMod} \overline{A}^o)$. $\square$

The following theorem justifies the definition of an ASF$^+$-regular $\mathbb{Z}$-algebra.

**Theorem 4.13.** *Let $A$ be an* Ext*-finite connected graded algebra. Then $A$ is an ASF-regular algebra of dimension $d \geq 1$ and of Gorenstein parameter $\ell$ if and only if $\overline{A}$ is an ASF$^+$-regular $\mathbb{Z}$-algebra of dimension $d \geq 1$ and of Gorenstein parameter $\ell$.*

*Proof.* If $\Phi : \operatorname{GrMod} \overline{A} \cong \operatorname{GrMod} A$ is an equivalence functor defined in Lemma 2.17, then $\Phi(k(-i)) \cong S_i$, so

$$\operatorname{pd}_{\overline{A}} S_i = \operatorname{pd}_A k(-i) = \operatorname{gldim} A$$

for every $i \in \mathbb{Z}$.

Since $\Phi^o(A(j-\ell)) \cong \overline{A} e_{j-\ell}$ in $\operatorname{GrMod} \overline{A}^o$ by Lemma 2.18 (1), and

$$\begin{aligned}(D \operatorname{R}\tau(\overline{A}))e_j &\cong D(e_j \operatorname{R}\tau(\overline{A})) \\ &\cong D\Phi(\operatorname{R}\tau(A(-j))) \\ &\cong \Phi^o D \operatorname{R}\tau(A(-j))\end{aligned}$$

in $\mathscr{D}(\operatorname{GrMod} \overline{A}^o)$ by Lemma 4.12 (3) and Lemma 2.18 (3),

$$(D \operatorname{R}\tau(\overline{A}))e_j \cong \overline{A} e_{j-\ell}[d]$$

for every $j \in \mathbb{Z}$ if and only if

$$D \operatorname{R}\tau(A(-j)) \cong A(j-\ell)[d]$$

in $\mathscr{D}(\operatorname{GrMod} A^o)$ for every $j \in \mathbb{Z}$ by Lemma 2.18 (1) if and only if

$$D \operatorname{R}\tau(A) \cong A(-\ell)[d]$$

in $\mathscr{D}(\operatorname{GrMod} A^o)$.

Similarly, since $\Phi(A(-i-\ell)) \cong e_{i+\ell}\overline{A}$ in $\operatorname{GrMod} \overline{A}$ by Lemma 2.17, and

$$e_i(D \operatorname{R}\tau(\overline{A})) \cong D(\operatorname{R}\tau(\overline{A})e_i) \cong D\Phi^o(\operatorname{R}\tau(A(i))) \cong \Phi D \operatorname{R}\tau(A(i))$$

in $\mathscr{D}(\operatorname{GrMod} \overline{A})$ by Lemma 4.12 (3) and Lemma 2.18 (3),

$$e_i(D \operatorname{R}\tau(\overline{A})) \cong e_{i+\ell}\overline{A}[d]$$

for every $i \in \mathbb{Z}$ if and only if $D \operatorname{R}\tau(A(i)) \cong A(-i-\ell)[d]$ in $\mathscr{D}(\operatorname{GrMod} A)$ for every $i \in \mathbb{Z}$ by Lemma 2.17 if and only if $D \operatorname{R}\tau(A) \cong A(-\ell)[d]$ in $\mathscr{D}(\operatorname{GrMod} A)$. $\square$

We have the following implication.

**Theorem 4.14.** *If $C$ is a right* Ext*-finite ASF$^+$-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$, then $C$ is an AS-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$.*

*Proof.* First note that in this situation, $\operatorname{cd}\tau < \infty$ by Lemma 3.22. Since

$$\begin{aligned}\operatorname{R}\operatorname{Hom}_C(S_i, P_j) &\cong \operatorname{R}\operatorname{Hom}_C(S_i, e_{j-\ell}(D \operatorname{R}\tau(C))[-d]) \\ &\cong \operatorname{R}\underline{\operatorname{Hom}}_C(S_i, D \operatorname{R}\tau(C))_{j-\ell}[-d] \\ &\cong D \operatorname{R}\tau(S_i)_{j-\ell}[-d] \\ &\cong (DS_i)_{j-\ell}[-d] \\ &\cong \begin{cases} k[-d] & \text{if } j = i+\ell \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

in $\mathscr{D}(\operatorname{Mod} k)$ by (ASF2$^+$) and Theorem 3.21, $C$ satisfies (ASR2). Since $\operatorname{pd} S_i \leq \operatorname{sgldim} C = d$ by (ASF1), while $\operatorname{pd} S_i \geq d$ by (ASR2) for every $i \in \mathbb{Z}$, $C$ satisfies (ASR1). □

4.3. **ASF$^{++}$-regular $\mathbb{Z}$-algebras.** We introduce another notion of regularity, which plays an essential role in this paper.

*Definition* 4.15. A locally finite connected $\mathbb{Z}$-algebra $C$ is called ASF$^{++}$-regular of dimension $d$ and of Gorenstein parameter $\ell$ if

(ASF1) $\operatorname{sgldim} C = d < \infty$, and
(ASF2$^{++}$) $D\,\mathrm{R}\,\tau(C) \cong C(0, -\ell)_\nu[d]$ in $\mathscr{D}(\operatorname{Bimod}(C - C))$ for some isomorphism $\nu : C \to C(-\ell)$ of $\mathbb{Z}$-algebras, called the Nakayama isomorphism of $C$.

*Remark* 4.16. Let $C$ be a $\mathbb{Z}$-algebra and $\lambda = \{\lambda_i \in C_{ii}\}_{i \in \mathbb{Z}}$ is a sequence of unit elements, then $I_\lambda : C \to C$ defined by $I_\lambda(a) = \lambda_i^{-1} a \lambda_j$ for $a \in C_{ij}$ is an automorphism of a $\mathbb{Z}$-algebra $C$, called an inner automorphism of $C$. If

$$\nu, \nu' : C \to C(-\ell)$$

are isomorphisms of $\mathbb{Z}$-algebras, and

$$\varphi : C(0, -\ell)_\nu \to C(0, -\ell)_{\nu'}$$

is an isomorphism of bigraded $C$-bimodules, then

$$\begin{aligned} \varphi(e_{i-\ell})\nu'(a) &= \varphi(e_{i-\ell}) * a = \varphi(e_{i-\ell} * a) = \varphi(e_{i-\ell}\nu(a)) \\ &= \varphi(\nu(a)) = \varphi(\nu(a)e_{j-\ell}) = \nu(a)\varphi(e_{j-\ell}) \end{aligned}$$

for $a \in (C(0, -\ell)_\nu)_{i, j+\ell} = C_{ij}$ so that $\nu(a), \nu'(a) \in C(-\ell)_{ij} = C_{i-\ell, j-\ell}$, so $\nu' = I_\lambda \circ \nu$ where $\lambda = \{\varphi(e_{i-\ell}) \in C(-\ell)_{ii}\}_{i \in \mathbb{Z}}$. It follows that the Nakayama isomorphism is unique up to inner automorphism of $C(-\ell)$ if it exists.

*Remark* 4.17. If $A$ is an AS-regular algebra, then $D\,\mathrm{R}\,\tau(A) \cong A(-\ell)_\nu[d]$ in $\mathscr{D}(\operatorname{GrMod} A^e)$ for some graded algebra automorphism $\nu \in \operatorname{Aut} A$ called the Nakayama automorphism of $A$. We say that $A$ is Calabi-Yau if $\nu = \mathrm{id}$ (up to inner automorphism). For a non-Calabi-Yau AS-regular algebra $A$, it is often the case that there exists a Calabi-Yau AS-regular algebra $A'$ such that $\operatorname{GrMod} A \cong \operatorname{GrMod} A'$ (see [8, Theorem 1.1]). In this case, $\overline{A} \cong \overline{A'}$ as $\mathbb{Z}$-algebras by [21, Theorem 1.1], so the notion of Calabi-Yau does not make sense for an AS-regular $\mathbb{Z}$-algebra. (The Nakayama isomorphism for a $\mathbb{Z}$-algebra is never the identity.)

**Lemma 4.18.** *Let $C$ be a locally finite connected $\ell$-periodic $\mathbb{Z}$-algebra, and $M$ a bigraded $C$-bimodule. If $M \cong C(0, -\ell)$ as graded left $C$-modules and $M \cong C(\ell, 0)$ as graded right $C$-modules, then there exists an isomorphism of $\mathbb{Z}$-algebras $\nu : C \to C(-\ell)$ such that $M \cong C(0, -\ell)_\nu$ as bigraded $C$-bimodules.*

*Proof.* Let $\psi : M \to C(0, -\ell)$ be an isomorphism of graded left $C$-modules. Define a bigraded $k$-linear map $\nu : C \to C(-\ell)$ by $\nu(a) = \psi(\psi^{-1}(e_{i-\ell})a)$ for

$a \in C_{ij}$. For $a \in C_{ij}, b \in C_{jk}$,

$$\begin{aligned}\nu(a)\nu(b) &= \psi(\psi^{-1}(e_{i-\ell})a)\psi(\psi^{-1}(e_{j-\ell})b) \\ &= \psi(\psi(\psi^{-1}(e_{i-\ell})a)\psi^{-1}(e_{j-\ell})b) \\ &= \psi(\psi^{-1}(\psi(\psi^{-1}(e_{i-\ell})a))b) \\ &= \psi(\psi^{-1}(e_{i-\ell})ab) = \nu(ab),\end{aligned}$$

so $\nu : C \to C(-\ell)$ is a homomorphism of $\mathbb{Z}$-algebras.

Since $C$ is $\ell$-periodic, there exists an isomorphism $\phi : C \to C(-\ell)$ of $\mathbb{Z}$-algebras. By Lemma 2.14 (3), there exists an isomorphism

$$\varphi : M \to C(\ell, 0) \to C(0, -\ell)_\phi$$

of graded right $C$-modules. If $\nu(a) = 0$, then $\psi^{-1}(e_{i-\ell})a = 0$, so

$$\varphi(\psi^{-1}(e_{i-\ell}))\phi(a) = \varphi(\psi^{-1}(e_{i-\ell})) * a = \varphi(\psi^{-1}(e_{i-\ell})a) = 0.$$

Since $0 \neq \varphi(\psi^{-1}(e_{i-\ell})) \in C_{i-\ell} \cong k$, $\phi(a) = 0$, so $a = 0$, hence $\nu$ is injective. Since $C$ is locally finite, $\nu$ is surjective, so $\nu : C \to C(-\ell)$ is an isomorphism of $\mathbb{Z}$-algebras.

We now consider the bigraded $k$-linear map $\psi : M \to C(0, -\ell)_\nu$. For $a \in C_{ij}, m \in M_{jt}, b \in C_{ts}$,

$$\begin{aligned}a\psi(m) * b = a\psi(m)\nu(b) &= \psi(am)\psi(\psi^{-1}(e_{t-\ell})b) \\ &= \psi(\psi(am)\psi^{-1}(e_{t-\ell})b) = \psi(\psi^{-1}(\psi(am))b) \\ &= \psi(amb),\end{aligned}$$

so $\psi : M \to C(0, -\ell)_\nu$ is an isomorphism of bigraded $C$-bimodules. □

**Theorem 4.19.** *A locally finite connected $\mathbb{Z}$-algebra is ASF$^{++}$-regular of dimension $d$ and of Gorenstein parameter $\ell$ if and only if it is $\ell$-periodic ASF$^{+}$-regular of dimension $d$ and of Gorenstein parameter $\ell$.*

*Proof.* If $C$ is an ASF$^{++}$-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$ with the Nakayama isomorphism $\nu : C \to C(-\ell)$, then $C$ is clearly $\ell$-periodic. Since

$$D\,\mathrm{R}\,\tau(C) \cong C(0, -\ell)_\nu[d]$$

in $\mathscr{D}(\mathrm{Bimod}(C - C))$,

$$D\,\mathrm{R}\,\tau(C)e_j \cong C(0, -\ell)e_j[d] \cong Q_{j-\ell}[d]$$

in $\mathscr{D}(\mathrm{GrMod}\, C^o)$. Similarly, since

$$D\,\mathrm{R}\,\tau(C) \cong C(0, -\ell)_\nu[d] \cong {}_{\nu^{-1}}C(\ell, 0)[d]$$

in $\mathscr{D}(\mathrm{Bimod}(C - C))$ by Lemma 2.14 (3),

$$e_i D\,\mathrm{R}\,\tau(C) \cong e_i C(\ell, 0)[d] \cong P_{i+\ell}[d]$$

in $\mathscr{D}(\mathrm{GrMod}\, C)$, so $C$ is ASF$^{+}$-regular.

Conversely, let $C$ be an $\ell$-periodic ASF$^{+}$-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$. By Remark 4.10 (2), $D\,\mathrm{R}^d\,\tau(C) \cong C(0, -\ell)$

as graded left $C$-modules and $D\,\mathrm{R}^d\,\tau(C) \cong C(\ell, 0)$ as graded right $C$-modules. By Lemma 4.18,

$$D\,\mathrm{R}^q\,\tau(C) \cong \begin{cases} C(0,-\ell)_\nu & \text{if } q = d \\ 0 & \text{if } q \neq d \end{cases}$$

for some isomorphism $\nu : C \to C(-\ell)$, so $C$ is ASF$^{++}$-regular. $\square$

In summary, for a right Ext-finite connected $\mathbb{Z}$-algebra $C$, we have the following implications by Theorem 4.8, Theorem 4.14, and Theorem 4.19:

$$\mathrm{ASF}^{++} \quad\Rightarrow\quad \mathrm{ASF}^{+} \quad\Rightarrow\quad \mathrm{AS} \quad\Rightarrow\quad \mathrm{ASF}$$

Moreover, if $C$ has a "balanced dualizing complex", then AS $\Leftrightarrow$ ASF by [13, Theorem 7.10], and if $C$ is $\ell$-periodic, then ASF$^{++}$ $\Leftrightarrow$ ASF$^{+}$ by Theorem 4.19.

## 5. C-construction

The C-construction defined below is essential to study $\mathbb{Z}$-algebras. We collect some properties of the C-construction which will be needed in this paper (see [12]).

### 5.1. **C-construction.**

*Definition* 5.1. Let $\mathscr{C}$ be a $k$-linear category.

(1) (*B-construction*) For $\mathcal{O} \in \mathscr{C}$ and $s \in \mathrm{Aut}_k \mathscr{C}$, we define a graded algebra $B(\mathscr{C}, \mathcal{O}, s) := \oplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(\mathcal{O}, s^i\mathcal{O})$.

(2) (*C-construction*) For a sequence of objects $\{E_i\}_{i\in\mathbb{Z}}$ in $\mathscr{C}$, we define a $\mathbb{Z}$-algebra $C(\mathscr{C}, \{E_i\}_{i\in\mathbb{Z}}) := \oplus_{i,j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(E_{-j}, E_{-i})$.

The following three lemmas are easy to prove, so we omit their proofs.

**Lemma 5.2.** [12, Lemma 3.3] *Let $\mathscr{C}$ be a $k$-linear abelian category. For $\mathcal{O} \in \mathscr{C}$ and $s \in \mathrm{Aut}_k \mathscr{C}$, $C(\mathscr{C}, \{s^i\mathcal{O}\}_{i\in\mathbb{Z}}) \cong \overline{B(\mathscr{C}, \mathcal{O}, s)}$ as $\mathbb{Z}$-algebras so that*

$$\mathrm{GrMod}\, C(\mathscr{C}, \{s^i\mathcal{O}\}_{i\in\mathbb{Z}}) \cong \mathrm{GrMod}\, B(\mathscr{C}, \mathcal{O}, s).$$

**Lemma 5.3.** *If $C$ is a $\mathbb{Z}$-algebra, then $C(r) \cong C(\mathrm{GrMod}\, C, \{P_{-i-r}\}_{i\in\mathbb{Z}})$ for every $r \in \mathbb{Z}$.*

**Lemma 5.4.** [5, Lemma 2.5] *Let $\{E_i\}_{i\in\mathbb{Z}}$ and $\{E'_i\}_{i\in\mathbb{Z}}$ be sequences in $k$-linear categories $\mathscr{C}$ and $\mathscr{C}'$, respectively.*

(1) *A functor $F : \mathscr{C} \to \mathscr{C}'$ induces a $\mathbb{Z}$-algebra homomorphism*

$$C(\mathscr{C}, \{E_i\}_{i\in\mathbb{Z}}) \to C(\mathscr{C}', \{F(E_i)\}_{i\in\mathbb{Z}}).$$

(2) *If there exists a fully faithful functor $F : \mathscr{C} \to \mathscr{C}'$ such that $F(E_i) \cong E'_i$ for every $i \in \mathbb{Z}$, then $C(\mathscr{C}, \{E_i\}_{i\in\mathbb{Z}}) \cong C(\mathscr{C}', \{E'_i\}_{i\in\mathbb{Z}})$ as $\mathbb{Z}$-algebras.*

*Remark* 5.5. For a $\mathbb{Z}$-algebra $C$, the quotient functor $\pi : \mathrm{GrMod}\, C \to \mathrm{Tails}\, C$ induces a homomorphism

$$\begin{aligned} C &\cong \underline{\mathrm{Hom}}_C(C, C) \\ &= C(\mathrm{GrMod}\, C, \{P_{-i}\}_{i\in\mathbb{Z}}) \\ &\to C(\mathrm{Tails}\, C, \{\mathcal{P}_{-i}\}_{i\in\mathbb{Z}}) \\ &= \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \end{aligned}$$

of $\mathbb{Z}$-algebras by Lemma 5.3 and Lemma 5.4, so $C(\mathrm{Tails}\, C, \{\mathcal{P}_{-i}\}_{i\in\mathbb{Z}})$ has a bigraded $C$-bimodule structure. On the other hand, $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$ has a bigraded $C$-bimodule structure as Definition 3.16, and we can see that

$$C(\mathrm{Tails}\, C, \{\mathcal{P}_{-i}\}_{i\in\mathbb{Z}}) = \underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C})$$

as bigraded $C$-bimodules.

5.2. **Ampleness.** The following notion of ampleness is a reindexed version of that introduced in [19].

*Definition* 5.6. We say that a sequence of objects $\{E_i\}_{i\in\mathbb{Z}}$ in an abelian category $\mathscr{C}$ is *ample* if

(A1) for every $X \in \mathscr{C}$ and every $m \in \mathbb{Z}$, there exists a surjection $\oplus_{j=1}^{s} E_{-i_j} \to X$ in $\mathscr{C}$ for some $i_1, \dots, i_s \geq m$, and
(A2) for every surjection $X \to Y$ in $\mathscr{C}$, there exists $m \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathscr{C}}(E_{-i}, X) \to \mathrm{Hom}_{\mathscr{C}}(E_{-i}, Y)$ is surjective for every $i \geq m$.

*Remark* 5.7. For an abelian category $\mathscr{C}$, $\mathcal{O} \in \mathscr{C}$ and $s \in \mathrm{Aut}\, \mathscr{C}$, the pair $(\mathcal{O}, s)$ is ample for $\mathscr{C}$ in the sense of [2] if and only if the sequence $\{s^i\mathcal{O}\}_{i\in\mathbb{Z}}$ is ample for $\mathscr{C}$ in the above sense.

**Theorem 5.8.** *Let $\mathscr{C}$ be a* Hom*-finite $k$-linear abelian category. If $\{E_i\}_{i\in\mathbb{Z}}$ is an ample sequence for $\mathscr{C}$ such that $\mathrm{End}_{\mathscr{C}}(E_i) = k$ for every $i \in \mathbb{Z}$, then $C := C(\mathscr{C}, \{E_i\})_{\geq 0}$ is a right coherent connected $\mathbb{Z}$-algebra and there exists an equivalence functor $F : \mathscr{C} \to \mathrm{tails}\, C$ such that $F(E_{-i}) \cong \mathcal{P}_i$ for every $i \in \mathbb{Z}$.*

*Proof.* This follows from [17, Theorem 2.3], which is just a re-indexed version of [19, Theorem 2.4]. ∎

5.3. $\chi$**-condition.**

*Definition* 5.9. Let $C$ be a right coherent $\mathbb{Z}$-algebra. We say that $C$ *satisfies* $\chi_i$ if $\mathrm{R}^q\,\tau(M)$ is right bounded for every $M \in \mathrm{grmod}\, C$ and every $q \leq i$. We say that $C$ *satisfies* $\chi$ if $C$ satisfies $\chi_i$ for every $i \in \mathbb{N}$.

*Remark* 5.10. Let $C$ be a right Ext-finite connected $\mathbb{Z}$-algebra. By Lemma 3.12, $C$ satisfies $\chi_1$ if and only if, for every $M \in \mathrm{grmod}\, C$, there exists $m \in \mathbb{Z}$ such that the canonical map $M_{\geq m} \to (\omega\pi M)_{\geq m}$ is an isomorphism.

The following lemma answers [19, Remarks, page 70].

**Lemma 5.11.** *If $C$ is a right coherent connected $\mathbb{Z}$-algebra satisfying $\chi_1$, then* tails $C$ *is* Hom*-finite and $\{\mathcal{P}_{-i}\}_{i\in\mathbb{Z}}$ is an ample sequence for* tails $C$.

*Proof.* (A1): For every $M \in \operatorname{grmod} C$, there exists a surjection $F \to M$ in $\operatorname{grmod} C$ where $F \in \operatorname{add}\{P_j\}_{j\in\mathbb{Z}}$ which induces a surjection $\mathcal{F} \to \mathcal{M}$ in $\operatorname{tails} C$ where $\mathcal{F} \in \operatorname{add}\{\mathcal{P}_j\}_{j\in\mathbb{Z}}$, so it is enough to show the condition (A1) for $\mathcal{P}_i$ for every $i \in \mathbb{Z}$. Since $C$ is right Ext-finite, for every $i \in \mathbb{Z}$, there exists an exact sequence $F \to P_i \to S_i \to 0$ in $\operatorname{grmod} C$ where $F \in \operatorname{add}\{P_j\}_{j>i}$ which induces a surjection $\mathcal{F} \to \mathcal{P}_i$ in $\operatorname{tails} C$ where $\mathcal{F} \in \operatorname{add}\{\mathcal{P}_j\}_{j>i}$. Applying this argument finite number of times, for every $m \in \mathbb{Z}$, there exists a surjection $\mathcal{F} \to \mathcal{P}_i$ in $\operatorname{tails} C$ where $\mathcal{F} \in \operatorname{add}\{\mathcal{P}_j\}_{j\geq m}$.

(A2): Every surjection $\pi\phi : \pi M \to \pi N$ in $\operatorname{tails} C$ where $M, N \in \operatorname{grmod} C$ is induced by a homomorphism $\phi : M \to N$ such that $\phi_{\geq m_1} : M_{\geq m_1} \to N_{\geq m_1}$ is a surjection in $\operatorname{grmod} C$ for some $m_1 \in \mathbb{Z}$. Since $C$ satisfies $\chi_1$, there exists $m_2, m_3 \in \mathbb{Z}$ such that $M_{\geq m_2} \cong (\omega\pi M)_{\geq m_2}$ and $N_{\geq m_3} \cong (\omega\pi N)_{\geq m_3}$ by Remark 5.10. By Lemma 2.2 (1), we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(\pi P_i, \pi M) \arrow[r, "\operatorname{Hom}_{\mathcal{C}}(\pi P_i, \pi\phi)"] \arrow[d, "\cong"'] & \operatorname{Hom}_{\mathcal{C}}(\pi P_i, \pi N) \arrow[d, "\cong"] \\
\operatorname{Hom}_{C}(P_i, \omega\pi M) \arrow[r] \arrow[d, "\cong"'] & \operatorname{Hom}_{C}(P_i, \omega\pi N) \arrow[d, "\cong"] \\
(\omega\pi M)_i \arrow[r] & (\omega\pi N)_i \\
M_i \arrow[u] \arrow[r, "\phi_i"] & N_i, \arrow[u]
\end{tikzcd}$$

so $\operatorname{Hom}_{\mathcal{C}}(\pi P_i, \pi\phi) : \operatorname{Hom}_{\mathcal{C}}(\pi P_i, \pi M) \to \operatorname{Hom}_{\mathcal{C}}(\pi P_i, \pi N)$ is surjective for every $i \geq \max\{m_1, m_2, m_3\}$.

We finally prove that $\operatorname{tails} C$ is Hom-finite. Let $\mathcal{M} = \pi M, \mathcal{N} = \pi N \in \operatorname{tails} C$ where $M, N \in \operatorname{grmod} C$. By Remark 5.10, there exists $m \in \mathbb{Z}$ such that $N_{\geq m} \cong (\omega\pi N)_{\geq m}$. By (A1), there exists a surjection $\oplus_{j=1}^s \mathcal{P}_{i_j} \to \mathcal{M}$ where $i_j \geq m$ for every $j = 1, \dots, s$. By Lemma 2.2 (1),

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \subset \operatorname{Hom}_{\mathcal{C}}(\oplus_{j=1}^s \pi P_{i_j}, \pi N) &\cong \oplus_{j=1}^s \operatorname{Hom}_C(P_{i_j}, \omega\pi N) \\
&= \oplus_{j=1}^s (\omega\pi N)_{i_j} = \oplus_{j=1}^s N_{i_j}.
\end{aligned}$$

Since $C$ is locally finite by Lemma 3.5, $N$ is locally finite, so $\operatorname{Hom}_{\mathcal{C}}(\pi M, \pi N)$ is finite dimensional. $\square$

**Lemma 5.12.** *If $C$ is a right coherent ASF-regular $\mathbb{Z}$-algebra of dimension $d \geq 1$, then $C$ satisfies $\chi$, so* $\operatorname{tails} C$ *is* Hom*-finite and $\{\mathcal{P}_{-i}\}_{i\in\mathbb{Z}}$ is an ample sequence for* $\operatorname{tails} C$.

*Proof.* Since $C$ is right coherent, for every $M \in \operatorname{grmod} C$, there exist $F \in \operatorname{add}\{P_i\}_{i\in\mathbb{Z}}$ and $L \in \operatorname{grmod} C$ such that $0 \to L \to F \to M \to 0$ is an exact sequence, which induces an exact sequence $\mathrm{R}^q\,\tau(F) \to \mathrm{R}^q(M) \to \mathrm{R}^{q+1}\,\tau(L)$ in $\operatorname{GrMod} C$ for every $q \in \mathbb{Z}$. Since $C$ is ASF-regular, $\mathrm{R}^q\,\tau(P_i)$ is right bounded for every $i \in \mathbb{Z}$ and $q \in \mathbb{Z}$, so $\mathrm{R}^q\,\tau(F)$ is right bounded for every $q \in \mathbb{Z}$.

Since $\mathrm{pd}(M) \leq d$ by [13, Proposition 4.11], $\mathrm{R}^q\,\tau(M) = \mathrm{R}^q\,\tau(L) = 0$ are right bounded for every $q > d$. In particular, $\mathrm{R}^{d+1}\,\tau(L) = 0$, so $\mathrm{R}^d\,\tau(M)$ is right bounded. By the same argument, $\mathrm{R}^d\,\tau(L)$ is right bounded, so $\mathrm{R}^{d-1}\,\tau(M)$ is right bounded. By induction, $\mathrm{R}^q\,\tau(M)$ is right bounded for every $q \in \mathbb{Z}$, so $C$ satisfies $\chi$. By Lemma 5.11, $\operatorname{tails} C$ is Hom-finite and $\{\mathcal{P}_{-i}\}_{i\in\mathbb{Z}}$ is an ample sequence for $\operatorname{tails} C$. □

## 5.4. **Quasi-Veronese $\mathbb{Z}$-algebras.**

*Definition* 5.13. Let $I_j := \{i \in \mathbb{Z} \mid jr \leq i \leq (j+1)r - 1\}$ for each $j \in \mathbb{Z}$. For a $\mathbb{Z}$-algebra $C$ and $r \in \mathbb{N}^+$, the *$r$-th quasi-Veronese $\mathbb{Z}$-algebra* $C^{[r]}$ of $C$ is defined by $(C^{[r]})_{st} := \oplus_{-i\in I_{-s}, -j\in I_{-t}} C_{ij}$

Note that if $C = C(\mathscr{C}, \{E_i\}_{i\in\mathbb{Z}})$, then $C^{[r]} = C(\mathscr{C}, \{\oplus_{i\in I_j} E_i\}_{j\in\mathbb{Z}})$.

**Lemma 5.14.** *For a $\mathbb{Z}$-algebra $C$ and $r \in \mathbb{N}^+$,* $\operatorname{GrMod} C \cong \operatorname{GrMod} C^{[r]}$.

*Proof.* Since $C^{[r]} \cong C$ as algebras (not as $\mathbb{Z}$-algebras unless $r = 1$) by a variant of [12, Corollary 3.6], the result holds by Lemma 2.13. □

**Lemma 5.15.** *A sequence of objects $\{E_i\}_{i\in\mathbb{Z}}$ in an abelian category $\mathscr{C}$ is ample if and only if $\{\oplus_{i\in I_j} E_i\}_{j\in\mathbb{Z}}$ is ample.*

*Proof.* (A1) Let $M \in \mathscr{C}$. If $\{E_i\}_{i\in\mathbb{Z}}$ is ample, then, for every $m \in \mathbb{Z}$, there exists a surjection $\oplus_{q=1}^s E_{-i_q} \to M$ in $\mathscr{C}$ for some $i_1, \dots, i_s \geq rm - (r-1)$. Since, for $1 \leq q \leq s$, $i_q \geq rm - (r-1)$, we have $-i_q \leq -rm + (r-1)$, and thus there exists $\ell_q \in \mathbb{Z}$ such that $\ell_q \geq m$ and

$$-r\ell_q \leq -i_q \leq -r\ell_q + (r-1).$$

Thus, $-i_q \in I_{-\ell_q}$ and therefore there exists a surjection

$$\bigoplus_{q=1}^{s} \bigoplus_{i\in I_{-\ell_q}} E_i \to M.$$

Conversely, if $\{\oplus_{i\in I_j} E_i\}_{j\in\mathbb{Z}}$ is ample, then, for every $m \in \mathbb{Z}$, there exists a surjection $\oplus_{q=1}^s(\oplus_{i\in I_{-j_q}} E_i) \to M$ in $\mathscr{C}$ for some $j_1, \dots, j_s \geq (m + r - 1)/r$. Thus, for $1 \leq q \leq s$, $rj_q \geq m + (r-1)$ and thus $-rj_q + (r-1) \leq -m$. It follows that if $i \in I_{-j_q}$, then $-i \geq m$, and so $\{E_i\}_{i\in\mathbb{Z}}$ is ample.

(A2) Let $M \to N$ be a surjection in $\mathscr{C}$. If $\{E_i\}_{i\in\mathbb{Z}}$ is ample, then there exists $m \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathscr{C}}(E_{-i}, M) \to \mathrm{Hom}_{\mathscr{C}}(E_{-i}, N)$ is surjective for every $i \geq m$. Now suppose $\ell \geq (m+r-1)/r$. Then $r\ell - (r-1) \geq m$ and thus if $w \in I_{-\ell}$ then $m \leq -w$. It follows that $\mathrm{Hom}_{\mathscr{C}}(\oplus_{i\in I_{-\ell}} E_i, M) \to \mathrm{Hom}_{\mathscr{C}}(\oplus_{i\in I_{-\ell}} E_i, N)$ is surjective for every $\ell \geq (m + r - 1)/r$.

Conversely, if $\{\oplus_{i\in I_j} E_i\}_{j\in\mathbb{Z}}$ is ample, then there exists $m \in \mathbb{Z}$ such that $\mathrm{Hom}_{\mathscr{C}}(\oplus_{i\in I_{-j}} E_i, M) \to \mathrm{Hom}_{\mathscr{C}}(\oplus_{i\in I_{-j}} E_i, N)$ is surjective for every $j \geq m$.

Now suppose $-i \geq mr - (r-1)$. Then $i \leq -mr + (r-1)$ and thus $i \in I_{-j}$ for some $j \geq m$ so that $\mathrm{Hom}_{\mathscr{C}}(E_i, M) \to \mathrm{Hom}_{\mathscr{C}}(E_i, N)$ is surjective. □

5.5. **Helices.** In this subsection, we recall some notions from [15].

*Definition* 5.16. Let $\mathscr{C}$ be a Hom-finite $k$-linear category. A *Serre functor for* $\mathscr{C}$ is a $k$-linear autoequivalence $S \in \operatorname{Aut}_k \mathscr{C}$ such that there exists a bifunctorial isomorphism

$$F_{X,Y} : \operatorname{Hom}_{\mathscr{C}}(X, Y) \to D \operatorname{Hom}_{\mathscr{C}}(Y, S(X))$$

for $X, Y \in \mathscr{C}$.

*Remark* 5.17. We explain the functoriality of a Serre functor $S$ in $Y$ in the above definition. (See [15, Remark 3.2] for the functoriality in $X$.) Define functors $G = \operatorname{Hom}_{\mathscr{C}}(X, -)$ and $H = D \operatorname{Hom}_{\mathscr{C}}(-, S(X)) = \operatorname{Hom}_k(\operatorname{Hom}_{\mathscr{C}}(-, S(X)), k)$. Fix $\beta \in \operatorname{Hom}_{\mathscr{C}}(Y, Y')$. Then

$$G(\beta) : \operatorname{Hom}_{\mathscr{C}}(X, Y) \to \operatorname{Hom}_{\mathscr{C}}(X, Y')$$

is given by $(G(\beta))(\alpha) = \beta \circ \alpha$. On the other hand,

$$H(\beta) : \operatorname{Hom}_k(\operatorname{Hom}_{\mathscr{C}}(Y, S(X)), k) \to \operatorname{Hom}_k(\operatorname{Hom}_{\mathscr{C}}(Y', S(X)), k)$$

is given by $((H(\beta))(\phi))(\gamma) = \phi(\gamma \circ \beta)$ for $\gamma \in \operatorname{Hom}_{\mathscr{C}}(Y', S(X))$. By functoriality, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathscr{C}}(X, Y) \arrow[r, "G(\beta)"] \arrow[d, "F_{X,Y}"'] & \operatorname{Hom}_{\mathscr{C}}(X, Y') \arrow[d, "F_{X,Y'}"] \\
D \operatorname{Hom}_{\mathscr{C}}(Y, S(X)) \arrow[r, "H(\beta)"] & D \operatorname{Hom}_{\mathscr{C}}(Y', S(X)),
\end{tikzcd}$$

so, for $\alpha \in \operatorname{Hom}_{\mathscr{C}}(X, Y)$ and $\gamma \in \operatorname{Hom}_{\mathscr{C}}(Y', S'(X))$, we have

$$F_{X,Y'}(\beta \circ \alpha)(\gamma) = (F_{X,Y'}(G(\beta)(\alpha)))(\gamma) = (H(\beta)(F_{X,Y}(\alpha)))(\gamma) = F_{X,Y}(\alpha)(\gamma \circ \beta).$$

*Definition* 5.18. Let $\mathscr{C}$ be an abelian category. A *bimodule* $\mathcal{L}$ *over* $\mathscr{C}$ is an adjoint pair of functors from $\mathscr{C}$ to itself with the suggestive notation $\mathcal{L} = (- \otimes_{\mathscr{C}} \mathcal{L}, \mathcal{H}om_{\mathscr{C}}(\mathcal{L}, -))$. A bimodule $\mathcal{L}$ over $\mathscr{C}$ is *invertible* if $- \otimes_{\mathscr{C}} \mathcal{L}$ is an autoequivalence of $\mathscr{C}$. In this case, the inverse bimodule of $\mathcal{L}$ is defined by $\mathcal{L}^{-1} = (- \otimes_{\mathscr{C}} \mathcal{L}^{-1}, \mathcal{H}om_{\mathscr{C}}(\mathcal{L}^{-1}, -)) = (\mathcal{H}om_{\mathscr{C}}(\mathcal{L}, -), - \otimes_{\mathscr{C}} \mathcal{L})$.

*Definition* 5.19. Let $\mathscr{C}$ be a Hom-finite $k$-linear abelian category. A *canonical bimodule for* $\mathscr{C}$ is an invertible bimodule $\omega_{\mathscr{C}}$ over $\mathscr{C}$ such that, for some $n \in \mathbb{Z}$, the autoequivalence $- \otimes^{\mathbf{L}}_{\mathscr{C}} \omega_{\mathscr{C}}[n]$ of $\mathscr{D}^b(\mathscr{C})$ induced by $- \otimes_{\mathscr{C}} \omega_{\mathscr{C}}$ is a Serre functor for $\mathscr{D}^b(\mathscr{C})$.

*Definition* 5.20. Let $\mathscr{T}$ be a triangulated category. For a set of objects $\mathcal{E}$ in $\mathscr{T}$, we denote by $\langle \mathcal{E} \rangle$ the smallest full triangulated subcategory of $\mathscr{T}$ containing $\mathcal{E}$ closed under isomorphisms and direct summands. We say that $S$ classically generates $\mathscr{T}$ if $\langle \mathcal{E} \rangle = \mathscr{T}$.

*Definition* 5.21. Let $\mathscr{T}$ be a $k$-linear triangulated category, and $\mathcal{E} = \{E_0, \dots, E_{\ell-1}\}$ a sequence of objects in $\mathscr{T}$.

(1) $\mathcal{E}$ is called *exceptional* if
   (E1) $\operatorname{End}_{\mathscr{T}}(E_i) = k$ for every $i = 0, \dots, \ell - 1$,

(E2) $\operatorname{Hom}_{\mathscr{T}}(E_i, E_i[q]) = 0$ for every $q \neq 0$ and every $i = 0, \dots, \ell - 1$, and

(E3) $\operatorname{Hom}_{\mathscr{T}}(E_i, E_j[q]) = 0$ for every $q$ and every $0 \leq j < i \leq \ell - 1$.

(2) $\mathcal{E}$ is called *full* if $\langle \mathcal{E} \rangle = \mathscr{T}$.

*Definition* 5.22. Let $\mathscr{C}$ be a $k$-linear abelian category having the canonical bimodule $\omega_{\mathscr{C}}$, and $\mathcal{E} = \{E_i\}_{i \in \mathbb{Z}}$ a sequence of objects in $\mathscr{D}^b(\mathscr{C})$.

(1) $\mathcal{E}$ is called a *helix of period* $\ell$ if

(H1) $E_{i+\ell} \cong E_i \otimes^{\mathbf{L}}_{\mathscr{C}} \omega_{\mathscr{C}}^{-1}$ for every $i$,

(H2) $\operatorname{Ext}^q_{\mathscr{C}}(E_i, E_i) \cong \begin{cases} k & \text{if } q = 0, \\ 0 & \text{if } q \neq 0 \end{cases}$ for every $i$, and

(H3) $\operatorname{Ext}^q_{\mathscr{C}}(E_i, E_j) = 0$ for every $q$ and every $0 < i - j < \ell$ (or equivalentry every $j < i < j + \ell$).

(2) A helix $\mathcal{E}$ of period $\ell$ is called *geometric* if $\operatorname{Ext}^q_{\mathscr{C}}(E_i, E_j) = 0$ for every $q \neq 0$ and every $i \leq j$.

(3) A helix $\mathcal{E}$ of period $\ell$ is called *full* if $\langle E_i, \dots, E_{i+\ell-1} \rangle = \mathscr{D}^b(\mathscr{C})$ for every $i$.

*Remark* 5.23. Let $\mathscr{C}$ be a $k$-linear abelian category having the canonical bimodule $\omega_{\mathscr{C}}$, and $\mathcal{E} := \{E_i\}_{i \in \mathbb{Z}}$ a (full) helix of period $\ell$ for $\mathscr{D}^b(\mathscr{C})$.

(1) $\{E_i, \dots, E_{i+\ell-1}\}$ is a (full) exceptional sequence for $\mathscr{D}^b(\mathscr{C})$ for every $i$.

(2) If $\operatorname{gldim} \mathscr{C} = n$, then the Serre functor for $\mathscr{D}^b(\mathscr{C})$ is given by $- \otimes^{\mathbf{L}}_{\mathscr{C}} \omega_{\mathscr{C}}[n]$ by [15, Remark 3.5], so

$$\operatorname{Hom}_{\mathscr{C}}(E_i, E_j) \cong D \operatorname{Hom}_{\mathscr{C}}(E_j, E_i \otimes^{\mathbf{L}}_{\mathscr{C}} \omega_{\mathscr{C}}[n]) \cong D \operatorname{Ext}^n_{\mathscr{C}}(E_j, E_{i-\ell})$$

for every $i, j \in \mathbb{Z}$.

(3) The above definition of a helix is not exactly the same as the one defined by mutation in [3] (see [15, Remark 3.17]).

**Lemma 5.24.** *If $\mathscr{C}$ is a $k$-linear abelian category having the canonical bimodule $\omega_{\mathscr{C}}$, and $\{E_i\}_{i \in \mathbb{Z}}$ is a helix of period $\ell$ for $\mathscr{D}^b(\mathscr{C})$, then $C = C(\mathscr{C}, \{E_i\}_{i \in \mathbb{Z}})$ is $\ell$-periodic.*

*Proof.* Since $\omega_{\mathscr{C}}^{-1} : \mathscr{C} \to \mathscr{C}$ is an equivalence functor such that $E_i \otimes \omega_{\mathscr{C}}^{-1} \cong E_{i+\ell}$ for every $i \in \mathbb{Z}$, $C(\ell) = C(\mathscr{C}, \{E_{i+\ell}\}_{i \in \mathbb{Z}}) \cong C(\mathscr{C}, \{E_i\}_{i \in \mathbb{Z}}) = C$ as $\mathbb{Z}$-algebras by Lemma 5.4, so $C$ is $\ell$-periodic. □

## 6. A Categorical Characterization of Quantum Projective $\mathbb{Z}$-spaces

The main theorem of [15] is a categorical characterization of quantum projective spaces, which are defined to be the noncommutative projective schemes associated to AS-regular algebras. In this last section, we will give a $\mathbb{Z}$-algebra version of this result. As a biproduct, we give a family of non-trivial examples of AS-regular $\mathbb{Z}$-algebras, which are constructed from noncommutative quadric hypersurfaces.

### 6.1. A Categorical Characterization.

*Definition* 6.1. Let $\mathscr{C}$ be a Hom-finite $k$-linear abelian category. We say that $\mathscr{C}$ satisfies *(GH) of period* $\ell$ if

(GH1) $\mathscr{C}$ has a canonical bimodule $\omega_{\mathscr{C}}$, and
(GH2) $\mathscr{C}$ has an ample sequence $\{E_i\}_{i\in\mathbb{Z}}$ which is a full geometric helix of period $\ell$ for $\mathscr{D}^b(\mathscr{C})$.

Let $\mathscr{C}$ be a Hom-finite $k$-linear abelian category satisfying (GH) of period $\ell$. We write

- $T := E_0 \oplus \cdots \oplus E_{\ell-1} \in \mathscr{C}$,
- $R := \operatorname{End}_{\mathscr{C}}(T)$, and
- $\Pi R := B(\mathscr{C}, T, - \otimes_{\mathscr{C}} \omega_{\mathscr{C}}^{-1})_{\geq 0}$.

**Lemma 6.2.** *If $\mathscr{C}$ is a* Hom*-finite $k$-linear abelian category satisfying (GH) of period $\ell$, then* $\operatorname{gldim} \Pi R = \operatorname{gldim} \mathscr{C} + 1 < \infty$.

*Proof.* Since $\{T \otimes_{\mathscr{C}} (\omega_{\mathscr{C}}^{-1})^{\otimes j}\}_{j\in\mathbb{Z}} = \{\oplus_{i\in I_j} E_i\}_{j\in\mathbb{Z}}$ where

$$I_j = \{i \in \mathbb{Z} \mid j\ell \leq i \leq (j+1)\ell - 1\}$$

is ample for $\mathscr{C}$ by Lemma 5.15, $(T, - \otimes_{\mathscr{C}} \omega_{\mathscr{C}}^{-1})$ is ample for $\mathscr{C}$ in the sense of [2] by Remark 5.7. By [15, Lemma 3.18, Lemma 3.19, Theorem 3.11 (2)], $\operatorname{gldim} \Pi R = \operatorname{gldim} \mathscr{C} + 1 < \infty$. □

*Remark* 6.3. In the above lemma, $\{T \otimes_{\mathscr{C}} (\omega_{\mathscr{C}}^{-1})^{\otimes i}\}_{i\in\mathbb{Z}}$ is a "full geometric relative helix of period 1" for $\mathscr{D}^b(\mathscr{C})$ in the sense of [15, Definition 3.14] by [15, Lemma 3.18], so $T$ is a "regular tilting object" for $\mathscr{D}^b(\mathscr{C})$ in the sense of [15, Definition 3.9] by [15, Lemma 3.19]. In the literature, $R = \operatorname{End}_{\mathscr{C}}(T)$ is called a "Fano algebra", and $\Pi R := B(\mathscr{C}, T, - \otimes_{\mathscr{C}} \omega_{\mathscr{C}}^{-1})_{\geq 0}$ is called the "preprojective algebra" of $R$ (cf. [11]).

The following theorem is a generalization of [3, Theorem 4.2] and one direction of [15, Theorem 4.1]. Our definition of a helix is not the same as the one defined in [3], so that the Koszul condition plays no role in the theorem below (see [15, Remark 3.17, Corollary 4.2]).

**Theorem 6.4.** *If $\mathscr{C}$ is a* Hom*-finite $k$-linear abelian category satisfying (GH) of period $\ell$, then $C := C(\mathscr{C}, \{E_i\}_{i\in\mathbb{Z}})_{\geq 0}$ is a right coherent ASF$^{++}$-regular $\mathbb{Z}$-algebra of dimension* $\operatorname{gldim} \mathscr{C} + 1$ *and of Gorenstein parameter $\ell$ such that $\mathscr{C} \cong \operatorname{tails} C$.*

*Proof.* Suppose that $\mathscr{C}$ satisfies (GH) of period $\ell$ with $n = \operatorname{gldim} \mathscr{C} < \infty$. Since $\mathscr{C}$ is Hom-finite, $C$ is locally finite. Since $C_{ii} = \operatorname{Hom}_{\mathscr{C}}(E_i, E_i) = k$ by (H2), $C$ is a connected $\mathbb{Z}$-algebra. By Theorem 5.8, $C$ is right coherent and $\mathscr{C} \cong \operatorname{tails} C$, so it is enough to show that $C$ is an $\ell$-periodic ASF$^{+}$-regular $\mathbb{Z}$-algebra of dimension $n+1$ and of Gorenstein parameter $\ell$ by Theorem 4.19. Let $T := E_0 \oplus \cdots \oplus E_{\ell-1} \in \mathscr{C}$ and $R := \operatorname{End}_{\mathscr{C}}(T)$.

First suppose that $n=0$. Since $\{E_0,\dots,E_{\ell-1}\}$ is an exceptional sequence, $R=\operatorname{End}_{\mathscr{C}}(\oplus_{i=0}^{\ell-1}E_i)=\begin{pmatrix} k & \operatorname{Hom}_{\mathscr{C}}(E_0,E_1) & \cdots & \operatorname{Hom}_{\mathscr{C}}(E_0,E_{\ell-1}) \\ 0 & k & \cdots & \operatorname{Hom}_{\mathscr{C}}(E_1,E_{\ell-1}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{pmatrix}$ is an upper triangular matrix algebra over $k$. By Remark 6.3 and [15, Theorem 3.10], $R$ is semisimple, so $R\cong k^{\ell}$, hence $\operatorname{Hom}_{\mathscr{C}}(E_i,E_j)=0$ for any $0\leq i\neq j\leq \ell-1$. Since $\mathcal{D}^b(\mathscr{C})\cong\mathcal{D}^b(\operatorname{mod} R)$, we have $\omega_{\mathscr{C}}=\operatorname{id}_{\mathscr{C}}$, so $E_{i+s\ell}\cong E_i$ for every $s\in\mathbb{Z}$. It follows that, for every $i,j\in\mathbb{Z}$, there exists $0\leq i',j'\leq \ell-1$ such that $E_i\cong E_{i'}, E_j\cong E_{j'}$, so

$$\begin{aligned} C(\mathscr{C},\{E_i\}_{i\in\mathbb{Z}})_{ij} &\cong \operatorname{Hom}_{\mathscr{C}}(E_{-j},E_{-i})\cong \operatorname{Hom}_{\mathscr{C}}(E_{-j'},E_{-i'}) \\ &\cong \begin{cases} k & \text{if } i'=j', \text{ or equivalently if } \ell\mid j-i \\ 0 & \text{if } i'\neq j', \text{ or equivalently if } \ell\nmid j-i, \end{cases} \\ &\cong k[x,x^{-1}]_{j-i}\cong \overline{k[x,x^{-1}]}_{ij} \end{aligned}$$

where $\deg x=\ell$. Since we have a canonical commutative diagram

$$\begin{array}{ccc} C(\mathscr{C},\{E_i\}_{i\in\mathbb{Z}})_{i,i+s\ell}\otimes C(\mathscr{C},\{E_i\}_{i\in\mathbb{Z}})_{i+s\ell,i+(s+t)\ell} & \longrightarrow & C(\mathscr{C},\{E_i\}_{i\in\mathbb{Z}})_{i,i+(s+t)\ell} \\ \cong\big\downarrow & & \big\downarrow\cong \\ \operatorname{Hom}_{\mathscr{C}}(E_{-(i+s\ell)},E_{-i})\otimes \operatorname{Hom}_{\mathscr{C}}(E_{-(i+(s+t)\ell)},E_{-(i+s\ell)}) & \longrightarrow & \operatorname{Hom}_{\mathscr{C}}(E_{-(i+(s+t)\ell)},E_{-i}) \\ \cong\big\downarrow & & \big\downarrow\cong \\ \operatorname{Hom}_{\mathscr{C}}(E_{-i},E_{-i})\otimes \operatorname{Hom}_{\mathscr{C}}(E_{-i},E_{-i}) & \longrightarrow & \operatorname{Hom}_{\mathscr{C}}(E_{-i},E_{-i}) \\ \cong\big\downarrow & & \big\downarrow\cong \\ k\otimes k & \longrightarrow & k \\ \cong\big\downarrow & & \big\downarrow\cong \\ kx^s\otimes kx^t & \longrightarrow & kx^{s+t} \\ \cong\big\downarrow & & \big\downarrow\cong \\ \overline{k[x,x^{-1}]}_{i,i+s\ell}\otimes \overline{k[x,x^{-1}]}_{i+s\ell,i+(s+t)\ell} & \longrightarrow & \overline{k[x,x^{-1}]}_{i,i+(s+t)\ell} \end{array}$$

$C:=C(\mathscr{C},\{E_i\}_{i\in\mathbb{Z}})_{\geq 0}\cong\overline{k[x,x^{-1}]}_{\geq 0}\cong\overline{k[x]}$ as $\mathbb{Z}$-algebras. Since $k[x]$ is an ASF-regular algebra of dimension 1 and of Gorenstein parameter $\ell$, $C$ is an $\ell$-periodic (in fact 1-periodic) ASF$^+$-regular $\mathbb{Z}$-algebra of dimension 1 and of Gorenstein parameter $\ell$ by Theorem 4.13.

Now suppose that $n\geq 1$. If $j<i<j+\ell$, then $\operatorname{Hom}_{\mathscr{C}}(E_i,E_j)=0$ by (H3). If $j+\ell\leq i$, then $\operatorname{Hom}_{\mathscr{C}}(E_i,E_j)\cong D\operatorname{Ext}^n_{\mathscr{C}}(E_j,E_{i-\ell})=0$ since $\{E_i\}$ is geometric and $n\geq 1$. It follows that $C=C(\mathscr{C},\{E_i\}_{i\in\mathbb{Z}})$, so $C$ is $\ell$-periodic by Lemma 5.24.

(ASF1): Since $E_i \otimes^{\mathbf{L}}_{\mathscr{C}} \omega_{\mathscr{C}}^{-1} \cong E_{i+\ell}$ for every $i \in \mathbb{Z}$ by (H1), and $\operatorname{Hom}_{\mathscr{C}}(E_i, E_j) = 0$ for every $j < i$ as shown above,

$$\Pi R := B(\mathscr{C}, T, - \otimes_{\mathscr{C}} \omega_{\mathscr{C}}^{-1})_{\geq 0} = B(\mathscr{C}, T, - \otimes_{\mathscr{C}} \omega_{\mathscr{C}}^{-1}).$$

Since $T \otimes_{\mathscr{C}} (\omega_{\mathscr{C}}^{-1})^{\otimes j} = \oplus_{i \in I_j} E_i$ where $I_j = \{i \in \mathbb{Z} \mid j\ell \leq i \leq (j+1)\ell - 1\}$,

$$\begin{aligned} \overline{\Pi R} &= \overline{B(\mathscr{C}, T, - \otimes_{\mathscr{C}} \omega_{\mathscr{C}}^{-1})} \\ &\cong C(\mathscr{D}^b(\mathscr{C}), \{T \otimes^{\mathbf{L}}_{\mathscr{C}} \omega_{\mathscr{C}}^{-j}\}_{j \in \mathbb{Z}}) \\ &\cong C(\mathscr{D}^b(\mathscr{C}), \{\oplus_{i \in I_j} E_i\}_{j \in \mathbb{Z}}) \\ &\cong C(\mathscr{C}, \{E_i\}_{i \in \mathbb{Z}})^{[\ell]} \\ &= C^{[\ell]}, \end{aligned}$$

so $\operatorname{GrMod} C \cong \operatorname{GrMod} C^{[\ell]} \cong \operatorname{GrMod} \overline{\Pi R} \cong \operatorname{GrMod} \Pi R$ by Lemma 5.14 and Lemma 2.17. Since $\{S_i\}_{i \in \mathbb{Z}}$ is the set of complete representatives of isomorphism classes of simple objects in $\operatorname{GrMod} C$,

$$\operatorname{sgldim} C = \operatorname{sgldim} \Pi R = \operatorname{gldim} \Pi R = n + 1$$

by [11, Proposition 2.7] and Lemma 6.2.

(ASF2), (ASF2$^+$): We will check the equivalent conditions given in Proposition 4.11. Since $\{E_i\}_{i \in \mathbb{Z}}$ is ample for $\mathscr{C}$, there exists an equivalence functor $\mathscr{C} \to \operatorname{tails} C$ sending $E_{-i}$ to $\mathcal{P}_i$ for every $i \in \mathbb{Z}$ by Theorem 5.8. Since $\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) = C(\operatorname{tails} C, \{\mathcal{P}_{-i}\}_{i \in \mathbb{Z}}) \cong C(\mathscr{C}, \{E_i\}_{i \in \mathbb{Z}}) =: C$ as bigraded $C$-bimodules by Lemma 5.4 and Remark 5.5, $e_i \underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) \cong e_i C = P_i$ as graded right $C$-modules for every $i \in \mathbb{Z}$ and $\underline{\operatorname{Hom}}_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) e_j \cong C e_j = Q_j$ as graded left $C$-modules for every $j \in \mathbb{Z}$.

If $i \leq j$, then $\operatorname{Ext}^q_{\mathcal{C}}(\mathcal{P}_j, \mathcal{P}_i) \cong \operatorname{Ext}^q_{\mathscr{C}}(E_{-j}, E_{-i}) = 0$ for all $q \neq 0$ since $\{E_i\}_{i \in \mathbb{Z}}$ is geometric. If $i - \ell < j < i$, then $\operatorname{Ext}^q_{\mathcal{C}}(\mathcal{P}_j, \mathcal{P}_i) \cong \operatorname{Ext}^q_{\mathscr{C}}(E_{-j}, E_{-i}) = 0$ for all $q$ by (H3). If $j \leq i - \ell$, then

$$\operatorname{Ext}^q_{\mathcal{C}}(\mathcal{P}_j, \mathcal{P}_i) \cong \operatorname{Ext}^q_{\mathscr{C}}(E_{-j}, E_{-i}) \cong D \operatorname{Ext}^{n-q}_{\mathscr{C}}(E_{-i}, E_{-j-\ell}) = 0$$

for all $q \neq n$ since $\{E_i\}_{i \in \mathbb{Z}}$ is geometric. It follows that

$$\underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{C}, \mathcal{C}) = \oplus_{i,j \in \mathbb{Z}} \operatorname{Ext}^q_{\mathcal{C}}(\mathcal{P}_j, \mathcal{P}_i) = 0$$

unless $q = 0, n$.

We will prove only (ASF2$^+$). The proof for (ASF2) is similar (see the proof of [15, Theorem 4.1]). Let $F_{ij}$ be the isomorphism

$$\begin{aligned} \operatorname{Ext}^{d-1}_{\mathscr{C}}(E_{-j}, E_{-i}) &\cong \operatorname{Hom}_{\mathscr{C}}(E_{-j}[1-d], E_{-i}) \\ &\to D \operatorname{Hom}_{\mathscr{C}}(E_{-i}, S(E_{-j}[1-d])) \\ &\cong D \operatorname{Hom}_{\mathscr{C}}(E_{-i}, E_{-j} \otimes^{\mathbf{L}}_{\mathscr{C}} \omega_{\mathscr{C}}) \\ &\cong D \operatorname{Hom}_{\mathscr{C}}(E_{-i}, E_{-j-\ell}) \end{aligned}$$

induced by the Serre functor, and consider the diagram

$$\begin{tikzcd}
C_{ij} \times \operatorname{Ext}^{d-1}_{\mathcal{C}}(\mathcal{C},\mathcal{C})_{jk} \arrow[r] \arrow[d] & \operatorname{Ext}^{d-1}_{\mathcal{C}}(\mathcal{C},\mathcal{C})_{ik} \arrow[d] \\
\operatorname{Hom}_{\mathscr{C}}(E_{-j},E_{-i}) \times \operatorname{Hom}_{\mathscr{C}}(E_{-k}[1-d],E_{-j}) \arrow[r, "\Phi"] \arrow[d, "\mathrm{id}\times F_{jk}"'] & \operatorname{Hom}_{\mathcal{C}}(E_{-k}[1-d],E_{-i}) \arrow[d, "F_{ik}"] \\
\operatorname{Hom}_{\mathscr{C}}(E_{-j},E_{-i}) \times D\operatorname{Hom}_{\mathscr{C}}(E_{-j},E_{-k-\ell}) \arrow[r, "\Psi"] \arrow[d] & D\operatorname{Hom}_{\mathcal{C}}(E_{-i},E_{-k-\ell}) \arrow[d] \\
C_{ij} \times D(C(\ell,0))_{jk} \arrow[r] & D(C(\ell,0))_{ik}.
\end{tikzcd}$$

The top square commutes since this is the way to define the graded left $C$-module structure for $\underline{\operatorname{Ext}}^{d-1}_{\mathcal{C}}(\mathcal{C},\mathcal{C})$ as in Definition 3.16 (2). The middle square commutes by the bifunctoriality of the Serre functor as follows: For $(\beta,\alpha) \in \operatorname{Hom}_{\mathscr{C}}(E_{-j},E_{-i}) \times \operatorname{Hom}_{\mathscr{C}}(E_{-k}[1-d],E_{-j})$, we have $\Phi(\beta,\alpha) = \beta\circ\alpha$. Moreover, for $(\beta,\phi) \in \operatorname{Hom}_{\mathscr{C}}(E_{-j},E_{-i}) \times D\operatorname{Hom}_{\mathscr{C}}(E_{-j},E_{-k-\ell})$, we have $\Psi(\beta,\phi)(\gamma) = \phi(\gamma\circ\beta)$ for every $\gamma \in \operatorname{Hom}_{\mathcal{C}}(E_{-i},E_{-k-\ell})$. By Remark 5.17,

$$\begin{aligned} F_{ik}(\Phi(\beta,\alpha))(\gamma) &= F_{ik}(\beta\circ\alpha)(\gamma) = F_{jk}(\alpha)(\gamma\circ\beta) \\ &= \Psi(\beta,F_{jk}(\alpha))(\gamma) = \Psi((\mathrm{id}\times F_{jk})(\beta,\alpha))(\gamma). \end{aligned}$$

The bottom square commutes since this is the way to define the graded left $C$-module structure for $D(C(\ell,0))$, so $\underline{\operatorname{Ext}}^{d-1}_{\mathcal{C}}(\mathcal{C},\mathcal{C}) \cong D(C(\ell,0))$ as graded left $C$-modules, hence (ASF2$^+$) holds. □

**Theorem 6.5.** *Let $\mathscr{C}$ be a $k$-linear abelian category. If $\mathscr{C}$ satisfies (GH), then the following are equivalent:*

(1) *$C := C(\mathscr{C},\{E_i\}_{i\in\mathbb{Z}})_{\geq 0}$ is right noetherian.*
(2) *$\mathscr{C}$ is a noetherian category.*
(3) *$E_i \in \mathscr{C}$ is a noetherian object for every $i \in \mathbb{Z}$.*

*Proof.* (1) ⇒ (2): If $C$ is right noetherian, then $\operatorname{grmod} C$ is a noetherian category by Lemma 2.10. It follows that, for every $\mathcal{M} \in \operatorname{tails} C$, there exists a noetherian object $M \in \operatorname{grmod} C$ such that $\mathcal{M} \cong \pi M \in \operatorname{tails} C$. Since $C$ is a right coherent connected $\mathbb{Z}$-algebra by Lemma 2.9, $\operatorname{Tors} C$ is a localizing subcategory of $\operatorname{GrMod} C$ by Lemma 3.7, so $\mathcal{M} \cong \pi M \in \operatorname{Tails} C$ is a noetherian object by [20, Lemma 5.8.3], hence $\operatorname{tails} C$ is a noetherian category.

(2) ⇒ (3): If $\mathscr{C}$ is a noetherian category, then $E_i \in \mathscr{C}$ is a noetherian object for every $i \in \mathbb{Z}$ by definition.

(3) ⇒ (1): If $E_i \in \mathscr{C}$ is a noetherian object for every $i \in \mathbb{Z}$, then $T := E_1 \oplus \cdots \oplus E_{\ell-1} \in \mathscr{C}$ is a noetherian object, so $\Pi R$ is right noetherian by [15, Theorem 4.1]. It follows that $\operatorname{grmod} C \cong \operatorname{grmod} \Pi R$ is a noetherian category, so $C$ is right noetherian by Lemma 2.10. □

We will next show the converse to Theorem 6.4 to complete our categorical characterization (a generalization of the other direction of [15, Theorem 4.1]).

**Theorem 6.6.** *Let $C$ be a right coherent ASF$^{++}$-regular $\mathbb{Z}$-algebra. If $X, Y \in \mathscr{D}^b(\operatorname{grmod} C)$, then we have a bifunctorial isomorphism*

$$\mathrm{R}\,\underline{\mathrm{Hom}}_C(I_0(X), I_0(Y)\underline{\otimes}_C^{\mathrm{L}} D\,\mathrm{R}\,Q(C)) \cong D\,\mathrm{R}\,\underline{\mathrm{Hom}}_C(I_0(Y), \mathrm{R}\,Q(I_0(X))).$$

*Proof.* By Lemma 3.26, $I_0(X)$ and $I_0(Y)$ are quasi-isomorphic to perfect complexes. Furthermore, since $X$ is bounded and $D\,\mathrm{R}\,Q(C)$ is bounded by Lemma 3.15, $D\,\mathrm{R}\,Q(I_0(X)) \in \mathscr{D}^b(\operatorname{Bimod}(C-K))$ by Theorem 3.29. We next note that by Lemma 3.12, there is a triangle

$$(D\,\mathrm{R}\,Q(X))_j \to (DX)_j \to (D\,\mathrm{R}\,\tau(X))_j$$

in $\mathscr{D}(\operatorname{Mod} k)$. Since, by Theorem 3.21, $D\,\mathrm{R}\,\tau(X)$ has locally finite homology, so does $D\,\mathrm{R}\,Q(X)$. It follows that $D\,\mathrm{R}\,Q(I_0(X))$ has locally finite homology. Therefore, by Proposition 3.27, Theorem 3.29 and [13, Corollary 6.2], we have bifunctorial isomorphisms

$$\begin{aligned}\mathrm{R}\,\underline{\mathrm{Hom}}_C(I_0(X), I_0(Y)\underline{\otimes}_C^{\mathrm{L}} D\,\mathrm{R}\,Q(C)) &\cong I_0(Y)\underline{\otimes}_C^{\mathrm{L}}\,\mathrm{R}\,\underline{\mathrm{Hom}}_C(I_0(X), D\,\mathrm{R}\,Q(C))\\ &\cong I_0(Y)\underline{\otimes}_C^{\mathrm{L}} D\,\mathrm{R}\,Q(I_0(X))\\ &\cong D\,\mathrm{R}\,\underline{\mathrm{Hom}}_C(I_0(Y), \mathrm{R}\,Q(I_0(X))),\end{aligned}$$

where in the last isomorphism, in order to apply [13, Corollary 6.2], we use [25, Proposition 3.1(1)], which holds since the last two expressions have locally finite homology by [7, Proposition 7.3(i)] and the fact that $I_0(Y)$ is quasi-isomorphic to a perfect complex and $\mathrm{R}\,Q(I_0(X)))$ has locally finite homology. □

**Lemma 6.7.** *If $C$ is a right coherent ASF$^{++}$-regular $\mathbb{Z}$-algebra and $X, Y \in \mathscr{D}^b(\operatorname{grmod} C)$, then*

1. *if $Y_{\geq n} = 0$ for some $n \in \mathbb{Z}$, then $\mathrm{R}\underline{\mathrm{Hom}}_C(X_{\geq n}, Y) = 0$, and*
2. $\mathrm{R}\underline{\mathrm{Hom}}_C(X_{\geq n}, \mathrm{R}\,Q(Y)) \cong \mathrm{R}\,\underline{\mathrm{Hom}}_C(X_{\geq n}, Y)$ *for some $n \in \mathbb{Z}$.*

*Proof.* (1) Since $C$ is right coherent, $X/X_{\geq n} \in \mathscr{D}^b(\operatorname{grmod} C)$, so $X_{\geq n} \in \mathscr{D}^b(\operatorname{grmod} C)$. If $F$ is the minimal free resolution of $X_{\geq n}$, then $F_{<n} = 0$, so

$$\mathrm{R}\underline{\mathrm{Hom}}_C(X_{\geq n}, Y) = \underline{\mathrm{Hom}}_C(F, Y) = 0,$$

hence the result.

(2) For $X, Y \in \mathscr{D}^b(\operatorname{grmod} C)$ and $n \in \mathbb{Z}$, we have a triangle $\mathrm{R}\,\tau(Y) \to Y \to \mathrm{R}\,Q(Y)$ in $\mathscr{D}(\operatorname{GrMod} C)$ by Lemma 3.12, which induces a triangle

$$\mathrm{R}\underline{\mathrm{Hom}}_C(X_{\geq n}, \mathrm{R}\,\tau(Y)) \to \mathrm{R}\underline{\mathrm{Hom}}_C(X_{\geq n}, Y) \to \mathrm{R}\underline{\mathrm{Hom}}_C(X_{\geq n}, \mathrm{R}\,Q(Y)).$$

Since $C$ is right coherent and $\operatorname{sgldim} C < \infty$, $Y \in \mathscr{D}^b(\operatorname{grmod} C)$ has a finitely generated free resolution $G$ of finite length by Lemma 3.26. Since $C$ is ASF$^{++}$-regular, it is ASF-regular and so $\mathrm{R}\,\tau(P_j) \cong (D(Q_{j-\ell}))[-d]$ is a complex whose terms are right bounded for every $j \in \mathbb{Z}$, so $\mathrm{R}\,\tau(G)_{\geq n} = 0$ for some $n \in \mathbb{Z}$, and so the result now follows from (1). □

The proof of the following result was inspired by the proof of [16, Appendix A].

**Theorem 6.8.** *Suppose that $C$ is a coherent ASF$^{++}$-regular $\mathbb{Z}$-algebra of dimension $d \geq 1$ and of Gorenstein parameter $\ell$ with Nakayama isomorphism $\nu : C \to C(-\ell)$. For $X, Y \in \mathscr{D}^b(\operatorname{grmod} C)$, there is a bifunctorial isomorphism*

$$\operatorname{Hom}_{\mathscr{D}^b(\operatorname{tails} C)}(\pi X, \pi(Y(-\ell))_\nu[d-1]) \cong D \operatorname{Hom}_{\mathscr{D}^b(\operatorname{tails} C)}(\pi Y, \pi X)$$

*i.e. the autoequivalence $S = (-)(-\ell)_\nu[d-1] : \mathscr{D}^b(\operatorname{tails} C) \to \mathscr{D}^b(\operatorname{tails} C)$ is a Serre functor.*

*Proof.* The fact that the equivalence $(-)(-\ell)_\nu : \operatorname{GrMod} C \to \operatorname{GrMod} C$ descends to an autoequivalence of $\operatorname{tails} C$ is a straightforward exercise using [22, Lemma 1.1]. It follows that there is an induced autoequivalence $S$ on $\mathscr{D}^b(\operatorname{tails} C)$ as in the statement of the theorem.

First note that, for $X \in \mathscr{D}^b(\operatorname{grmod} C)$ and for any $n \in \mathbb{Z}$, $X_{\geq n} \in \mathscr{D}^b(\operatorname{grmod} C)$ and $\pi X \cong \pi X_{\geq n}$. Let $\mathscr{D} := \mathscr{D}(\operatorname{GrMod} C)$ and $\mathscr{C} := \mathscr{D}(\operatorname{Tails} C)$. Since $\mathrm{R}\, Q \cong \mathrm{R}\, \omega \circ \pi : \mathscr{D} \to \mathscr{D}$ by the proof of [4, Lemma 4.1.6] and $(\pi, \mathrm{R}\,\omega)$ is an adjoint pair of functors between $\mathscr{D}$ and $\mathscr{C}$, we have bifunctorial isomorphisms

$$\operatorname{Hom}_{\mathscr{C}}(\pi X, \pi(Y(-\ell))_\nu[d-1]) \cong \operatorname{Hom}_{\mathscr{D}}(X_{\geq n}, \mathrm{R}\, Q(Y(-\ell))_\nu)[d-1]),$$

and

$$D \operatorname{Hom}_{\mathscr{C}}(\pi Y, \pi X) \cong D \operatorname{Hom}_{\mathscr{D}}(Y, \mathrm{R}\, Q(X_{\geq n}))$$

for any $n \in \mathbb{Z}$, so it is enough to show

$$D \operatorname{Hom}_{\mathscr{D}}(Y, \mathrm{R}\, Q(X_{\geq n})) \cong \operatorname{Hom}_{\mathscr{D}}(X_{\geq n}, \mathrm{R}\, Q(Y(-\ell))_\nu)[d-1])$$

for some $n \in \mathbb{Z}$ by Lemma 3.11.

By Theorem 6.6, we have a bifunctorial isomorphism

$$D \,\mathrm{R} \operatorname{Hom}_C(I_0(Y), \mathrm{R}\, Q I_0(X_{\geq n})) \cong \mathrm{R} \operatorname{Hom}_C(I_0(X_{\geq n}), I_0(Y) \underline{\otimes}_C^{\mathrm{L}} D \,\mathrm{R}\, Q(C)),$$

so, taking zeroeth cohomology on both sides,

$$D \operatorname{Hom}_{\mathscr{D}}(Y, \mathrm{R}\, Q(X_{\geq n})) \cong \operatorname{Hom}_{\mathscr{D}}(X_{\geq n}, Y \underline{\otimes}_C^{\mathrm{L}} D \,\mathrm{R}\, Q(C))$$

by Lemma 3.20 (3).

In order to compute this last expression, we first note that by Lemma 3.14, there is a triangle

$$DC[-1] \to D \,\mathrm{R}\, \tau(C)[-1] \cong C(0, -\ell)_\nu[d-1] \to D \,\mathrm{R}\, Q(C)$$

in $\mathscr{D}(\operatorname{Bimod}(C - C))$. Since $(Y \underline{\otimes}_C^{\mathrm{L}} DC[-1])_{\geq n} = 0$ for some $n \in \mathbb{Z}$, we have $\operatorname{Hom}_{\mathscr{D}}(X_{\geq n}, Y \underline{\otimes}_C^{\mathrm{L}} DC[-1]) = 0$ by Lemma 6.7 (1). Since $\operatorname{Hom}_{\mathscr{D}}(X_{\geq n}, -)$ is cohomological,

$$\operatorname{Hom}_{\mathscr{D}}(X_{\geq n}, Y \underline{\otimes}_C^{\mathrm{L}} D \,\mathrm{R}\, Q(C)) \cong \operatorname{Hom}_{\mathscr{D}}(X_{\geq n}, Y \underline{\otimes}_C^{\mathrm{L}} C(0, -\ell)_\nu[d-1])$$

for any $n \gg 0$. Finally, since

$$\begin{aligned} \operatorname{Hom}_{\mathscr{D}}(X_{\geq n}, Y \underline{\otimes}_C^{\mathrm{L}} C(0, -\ell)_\nu[d-1]) &\cong \operatorname{Hom}_{\mathscr{D}}(X_{\geq n}, Y(-\ell)_\nu[d-1]) \\ &\cong \operatorname{Hom}_{\mathscr{D}}(X_{\geq n}, \mathrm{R}\, Q(Y(-\ell)_\nu)[d-1]) \end{aligned}$$

for any $n \gg 0$ by Lemma 2.15 (3) and Lemma 6.7 (2), the result follows. $\square$

The following lemma is standard. We include the proof for the convenience of the reader.

**Lemma 6.9.** *Let $\mathscr{C}$ be an abelian category and $\mathcal{E}$ a set of objects in $\mathscr{C}$. If $X$ is a bounded complex such that $X^q \in \mathcal{E}$ for every $q \in \mathbb{Z}$, then $X \in \langle \mathcal{E} \rangle \subset \mathscr{D}^b(\mathscr{C})$.*

*Proof.* Since $\langle \mathcal{E} \rangle$ is closed under shifting complexes, we may assume that $X^q = 0$ for every $q < -n$ and $0 < q$. If $n = 0$, then the result is trivial. In general, since there is an exact triangle

$$X^{-n}[n-1] \to X^{\geq -n+1} \to X \to X^{-n}[n],$$

and $X^{-n}[n-1], X^{\geq -n+1} \in \langle \mathcal{E} \rangle$ by induction, we have $X \in \langle \mathcal{E} \rangle$. □

**Lemma 6.10.** *If $C$ is a right coherent AS-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$, then $\langle \mathcal{P}_j, \dots, \mathcal{P}_{j+\ell-1} \rangle = \mathscr{D}^b(\operatorname{tails} C)$ for every $j \in \mathbb{Z}$.*

*Proof.* If $C$ is a right coherent AS-regular $\mathbb{Z}$-algebra of dimension $d$ and of Gorenstein parameter $\ell$, then we have exact sequences

$$0 \to P_{j+\ell} \to F^{d-1} \to \cdots \to F^1 \to P_j \to S_j \to 0$$
$$0 \to P_{j+\ell-1} \to G^{d-1} \to \cdots \to G^1 \to P_{j-1} \to S_{j-1} \to 0$$

in $\operatorname{grmod} C$ where $F^s \in \operatorname{add}\{P_i\}_{j<i<j+\ell}$ for $1 \leq s \leq d-1$ and $G^t \in \operatorname{add}\{P_i\}_{j-1<i<j+\ell-1}$ for $1 \leq t \leq d-1$ by Remark 4.6, which induce exact sequences

$$0 \to \mathcal{P}_{j+\ell} \to \mathcal{F}^{d-1} \to \cdots \to \mathcal{F}^1 \to \mathcal{P}_j \to 0$$
$$0 \to \mathcal{P}_{j+\ell-1} \to \mathcal{G}^{d-1} \to \cdots \to \mathcal{G}^1 \to \mathcal{P}_{j-1} \to 0$$

in $\operatorname{tails} C$, so $\mathcal{P}_{j-1}, \mathcal{P}_{j+\ell} \in \langle \mathcal{P}_j, \dots, \mathcal{P}_{j+\ell-1} \rangle$ by Lemma 6.9. By induction, $\mathcal{P}_i \in \langle \mathcal{P}_j, \dots, \mathcal{P}_{j+\ell-1} \rangle$ for every $i \in \mathbb{Z}$. Since every $X \in \mathcal{D}^b(\operatorname{grmod} C)$ has a finitely generated free resolution of finite length by Lemma 3.26,

$$\langle \mathcal{P}_j, \dots, \mathcal{P}_{j+\ell-1} \rangle = \langle \{\mathcal{P}_i\}_{i \in \mathbb{Z}} \rangle = \mathcal{D}^b(\operatorname{tails} C)$$

by Lemma 6.9. □

The theorem below is the converse to Theorem 6.4, a generalization of the other direction of [15, Theorem 4.1]. (See also [12, Theorem 4.7]).

**Theorem 6.11.** *If $C$ is a right coherent ASF$^{++}$-regular $\mathbb{Z}$-algebra of dimension $d \geq 1$ and of Gorenstein parameter $\ell$ with the Nakayama isomorphism $\nu : C \to C(-\ell)$, then*

(GH1) *$(-) \otimes_{\mathcal{C}} \omega_{\mathcal{C}} := (-)(-\ell)_\nu$ is a canonical bimodule for $\operatorname{tails} C$, and*

(GH2) *$\{\mathcal{P}_{-i}\}_{i \in \mathbb{Z}}$ is an ample sequence for $\operatorname{tails} C$ which is a full geometric helix of period $\ell$ for $\mathcal{D}^b(\operatorname{tails} C)$,*

*so that $\operatorname{tails} C$ satisfies (GH) of period $\ell$.*

*Proof.* (GH1): This follows from Theorem 6.8.

(GH2): By Lemma 5.12, $\{\mathcal{P}_{-i}\}_{i \in \mathbb{Z}}$ is an ample sequence for $\operatorname{tails} C$. Since $C$ is $\ell$-periodic, $\mathcal{P}_{-j-\ell} \cong \mathcal{P}_{-j}(\ell)_{\nu^{-1}} \cong \mathcal{P}_{-j} \otimes^{\mathbf{L}}_{\mathcal{C}} \omega_{\mathcal{C}}^{-1}$ for every $j \in \mathbb{Z}$ by Lemma 2.16, so $\{\mathcal{P}_{-i}\}_{i\in\mathbb{Z}}$ satisfies (H1). Since $(D(C(0,-\ell)))_{ij} = D(C(0,-\ell)_{ji}) = D(C_{j,i-\ell})$, if $d=1$, then

$$\begin{aligned} \operatorname{Ext}^q_{\mathcal{C}}(\mathcal{P}_j,\mathcal{P}_i) &\cong \underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{C},\mathcal{C})_{ij} \\ &\cong \begin{cases} (C\oplus D(C(0,-\ell)))_{ij} = C_{ij}\oplus D(C_{j,i-\ell}) & \text{if } q=0 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

and, if $d\geq 2$, then

$$\operatorname{Ext}^q_{\mathcal{C}}(\mathcal{P}_j,\mathcal{P}_i) \cong \underline{\operatorname{Ext}}^q_{\mathcal{C}}(\mathcal{C},\mathcal{C})_{ij} \cong \begin{cases} C_{ij} & \text{if } q=0 \\ (D(C(0,-\ell)))_{ij} = D(C_{j,i-\ell}) & \text{if } q=d-1 \\ 0 & \text{otherwise} \end{cases}$$

by Proposition 4.11. Since $D(C_{i,i-\ell})=0$, we have

$$\operatorname{Ext}^q_{\mathcal{C}}(\mathcal{P}_i,\mathcal{P}_i) \cong \begin{cases} C_{ii}=k & \text{if } q=0, \\ 0 & \text{if } q\neq 0 \end{cases}$$

for every $i$, so $\{\mathcal{P}_{-i}\}_{i\in\mathbb{Z}}$ satisfies (H2). If $j<i$, then $C_{ij}=0$ and if $i<j+\ell$, then $D(C_{j,i-\ell})=0$, so, in either case, $\operatorname{Ext}^q_{\mathcal{C}}(\mathcal{P}_j,\mathcal{P}_i)=0$ for every $q$ and $j<i<j+\ell$, so $\{\mathcal{P}_{-i}\}_{i\in\mathbb{Z}}$ satisfies (H3). Moreover, $\operatorname{Ext}^q_{\mathcal{C}}(\mathcal{P}_j,\mathcal{P}_i)=0$ for every $q\neq 0$ and every $i\leq j$, so $\{\mathcal{P}_i\}_{i\in\mathbb{Z}}$ is a geometric helix of period $\ell$ for $\mathcal{D}^b(\operatorname{tails} C)$. By Lemma 6.10, $\{\mathcal{P}_i\}_{i\in\mathbb{Z}}$ is full. $\square$

6.2. **An Application to Noncommutative Quadric Hypersurfaces.** For the rest of the paper, we assume that $k$ is an algebraically closed field of characteristic 0. If $C$ is a 3-dimensional "cubic" AS-regular $\mathbb{Z}$-algebra $C$ in the sense of [26], which in particular implies that $C$ is a right noetherian AS-regular $\mathbb{Z}$-algebra of dimension 3 and of Gorenstein parameter 4 (cf. [26, Corollary 5.5.9]), then $\operatorname{tails} C$ is considered as a noncommutative $\mathbb{P}^1\times\mathbb{P}^1$. On the other hand, if $S$ is a 4-dimensional noetherian quadratic AS-regular algebra, and $f\in S_2$ is a regular normal element, then $\operatorname{tails} S/(f)$ is considered as a noncommutative quadric surface (see [23]). Since $\mathbb{P}^1\times\mathbb{P}^1$ is isomorphic to a smooth quadric in $\mathbb{P}^3$ in commutative algebraic geometry, we expect a similar result in noncommutative algebraic geometry. The following implication was known:

**Theorem 6.12.** [26, Corollary 6.7] *For every 3-dimensional "cubic" AS-regular $\mathbb{Z}$-algebra $C$, there exist a 4-dimensional right noetherian quadratic AS-regular algebra $S$ and a regular normal element $f\in S_2$ such that* $\operatorname{tails} C\cong \operatorname{tails} S/(f)$.

This paper gives a partial converse.

**Theorem 6.13.** *If $S$ is a 4-dimensional noetherian quadratic AS-regular algebra and $f\in S_2$ is a regular central element such that*

- *$S/(f)$ is a domain,*
- *$S/(f)$ is a noncommutative graded isolated singularity in the sense that* $\operatorname{gldim}(\operatorname{tails} S/(f))<\infty$ *(that is,* $\operatorname{tails} S/(f)$ *is "smooth"), and*
- *$S/(f)$ is "standard" in the sense of* [15, Section 5],

*then there exists a right noetherian AS-regular $\mathbb{Z}$-algebra $C$ of dimension 3 and of Gorenstein parameter 4 such that* $\operatorname{tails} S/(f) \cong \operatorname{tails} C$.

*Proof.* It is known that $\operatorname{gldim} S/(f) = 2$ (cf. [15, Section 5]). By [15, Theorem 5.16], there exists a full geometric helix $\{E_i\}_{i\in\mathbb{Z}}$ of period 4 for $\mathscr{D}(\operatorname{tails} S/(f))$. By the proof of [15, Theorem 5.17] and Lemma 5.15, $\{E_i\}_{i\in\mathbb{Z}}$ is ample, so $\operatorname{tails} S/(f)$ satisfies (GH) of period 4, hence there exists an AS-regular $\mathbb{Z}$-algebra $C$ of dimension 3 and of Gorenstein parameter 4 such that $\operatorname{tails} S/(f) \cong \operatorname{tails} C$ by Theorem 6.4. Since $S/(f)$ is right noetherian, $\operatorname{tails} S/(f)$ is a noetherian categoy, so $C$ is right noetherian by Theorem 6.5. □

We partially extend the above theorem to noncommutative quadric hypesurfaces below.

**Theorem 6.14.** *Let* $S := k\langle x_1, \dots, x_n\rangle/(x_ix_j - \epsilon_{ij}x_jx_i)$ *be a* $\pm 1$ *skew polynomial algebra where* $\epsilon_{ii} = 1$ *for every* $i$, $\epsilon_{ij} = \epsilon_{ji} = \pm 1$ *for every* $i \neq j$, *and* $A := S/(x_1^2+\cdots+x_n^2)$. *If* $n \geq 3$, *then there exists a right noetherian AS-regular $\mathbb{Z}$-algebra $C$ of dimension* $n-1$ *such that* $\operatorname{tails} A \cong \operatorname{tails} C$. *In particular, for every (commutative) smooth quadric hypersurface* $Q \subset \mathbb{P}^{n-1}$, *there exists a right noetherian AS-regular $\mathbb{Z}$-algebra $C$ of dimension* $n-1$ *and of Gorenstein parameter* $\ell = \begin{cases} n-1 & \text{if } n \text{ is odd,} \\ n & \text{if } n \text{ is even,} \end{cases}$ *such that* $\operatorname{tails} C \cong \operatorname{coh} Q$, *the category of coherent sheaves on* $Q$.

*Proof.* Since $A$ is right noetherian, $\operatorname{tails} A$ is a noetherian category, so it is enough to show that $\operatorname{tails} A$ satisfies (GH) by Theorem 6.4 and Theorem 6.5.

(GH1): It is known that $A$ is a noetherian AS-Gorenstein algebra of dimensiona $n-1 \geq 2$ (cf. [24, Section 2.1]) and of Gorenstein parameter $n-2 \geq 1$ and that $\operatorname{gldim}(\operatorname{tails} A) = n-2$ (cf. [24, Section 2.2]), so $\operatorname{tails} A$ has a canonical bimodule $\omega_{\mathcal{A}} = \mathcal{A}_\nu(-n+2)$ where $\nu$ is the Nakayama automorphism of $A$ (cf. [24, Lemma 2.2].)

(GH2): Let $\operatorname{Ind}^0(\operatorname{CM}^{\mathbb{Z}}(A)) = \{A, X_1, X_2, \dots, X_\alpha\}$ be the set of complete representatives of isomorphism classes of indecomposable graded maximal Cohen-Macaulay right $A$-modules generated in degree 0 where we say that $M \in \operatorname{grmod} A$ is a graded maximal Cohen-Macaulay if $\underline{\operatorname{Ext}}^q_A(M, A) = 0$ for every $q \neq 0$. We label the sequence

$$\mathcal{A}(-n+3), \dots, \mathcal{A}(-1), \mathcal{A}, \mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_\alpha$$

by $E_0, \dots, E_{\ell-1}$ where $\ell = n-2+\alpha$. We extend in both directions the sequence $E_0, \dots, E_{\ell-1}$ to $\{E_i\}_{i\in\mathbb{Z}}$ by $E_{i+r\ell} := E_i \otimes^{\mathbf{L}}_{\mathcal{A}} (\omega_{\mathcal{A}}^{-1})^{\otimes r}$ so that $\{E_i\}_{i\in\mathbb{Z}}$ satisfies (H1). By [24, Lemma 3.15] and Lemma 5.15, $\{E_i\}_{i\in\mathbb{Z}}$ is an ample sequence for $\operatorname{tails} A$.

If $X \in \operatorname{Ind}^0(\operatorname{CM}^{\mathbb{Z}}(A))$, then $X \otimes_A \omega_A^{\otimes r}(r(n-2)) \cong X_{\nu^r} \in \operatorname{Ind}^0(\operatorname{CM}^{\mathbb{Z}}(A))$ for every $r \in \mathbb{Z}$, so, for every $i \in \mathbb{Z}$, there exists $s \in \mathbb{Z}$ and $X \in \operatorname{Ind}^0(\operatorname{CM}^{\mathbb{Z}}(A))$ such that $E_i \cong \mathcal{X}(s)$. Since $\mathcal{A}(i) \otimes^{\mathbf{L}}_{\mathcal{A}} \omega_{\mathcal{A}}^{\otimes r} \cong \mathcal{A}_{\nu^r}(i-r(n-2)) \cong \mathcal{A}(i-r(n-2))$ for every $i \in \mathbb{Z}$, there exists a permutation $\sigma$ on $\{1, \dots, \alpha\}$ such that $\mathcal{X}_j \otimes^{\mathbf{L}}_{\mathcal{A}} \omega_{\mathcal{A}}^{\otimes r} \cong$

$\mathcal{X}_{\sigma^r(j)}(-r(n-2))$ (cf. the proof of [24, Lemma 3.15]). It follows that the sequence $\{E_i\}_{i\in\mathbb{Z}}$ looks like

$$\begin{aligned}
&\cdots, \mathcal{A}(r(n-2)-n+3), \dots, \mathcal{A}(r(n-2)-1), \mathcal{A}(r(n-2)),\\
&\quad \mathcal{X}_{\sigma^{-r}(1)}(r(n-2)), \mathcal{X}_{\sigma^{-r}(2)}(r(n-2)), \dots, \mathcal{X}_{\sigma^{-r}(\alpha)}(r(n-2)),\\
&\quad \mathcal{A}((r+1)(n-2)-n+3) = \mathcal{A}(r(n-2)+1), \dots, \mathcal{A}((r+1)(n-2)-1), \mathcal{A}((r+1)(n-2)),\\
&\quad \mathcal{X}_{\sigma^{-(r+1)}(1)}((r+1)(n-2)), \mathcal{X}_{\sigma^{-(r+1)}(2)}((r+1)(n-2)), \dots, \mathcal{X}_{\sigma^{-(r+1)}(\alpha)}((r+1)(n-2)), \ \cdots
\end{aligned}$$

Since $E_0, \dots, E_{\ell-1}$ is an exceptional sequence by [24, Lemma 3.12],

$$\operatorname{Ext}^q_{\mathcal{A}}(E_i, E_i) \cong \begin{cases} k & \text{if } q = 0,\\ 0 & \text{if } q \neq 0,\end{cases}$$

so $\{E_i\}_{i\in\mathbb{Z}}$ satisfies (H2).

Since $\operatorname{gldim}(\operatorname{tails} A) = n-2$, $\operatorname{Ext}^q_{\mathcal{A}} = 0$ for every $q \geq n-1$. For $A \not\cong X, Y \in \operatorname{Ind}^0(\operatorname{CM}^{\mathbb{Z}}(A))$,

$$\begin{aligned}
(1)\quad \operatorname{Ext}^q_{\mathcal{A}}(\mathcal{A}(s), \mathcal{A}(t)) &\cong \begin{cases} A_{t-s} & \text{if } q = 0,\\ 0 & \text{if } 1 \leq q \leq n-3,\\ D(A_{s-t-n+2}) & \text{if } q = n-2,\end{cases}\\
(2)\quad \operatorname{Ext}^q_{\mathcal{A}}(\mathcal{A}(s), \mathcal{X}(t)) &\cong \begin{cases} X_{t-s} & \text{if } q = 0,\\ 0 & \text{if } 1 \leq q \leq n-3,\\ D(\underline{\operatorname{Hom}}_A(X, A)_{s-t-n+2}) & \text{if } q = n-2,\end{cases}\\
(3)\quad \operatorname{Ext}^q_{\mathcal{A}}(\mathcal{X}(s), \mathcal{A}(t)) &\cong \begin{cases} \underline{\operatorname{Hom}}_A(X, A)_{t-s} & \text{if } q = 0,\\ 0 & \text{if } 0 \leq q \leq n-3,\\ D((X_\nu)_{s-t-n+2}) & \text{if } q = n-2,\end{cases}\\
(4)\quad \operatorname{Ext}^q_{\mathcal{A}}(\mathcal{X}(s), \mathcal{Y}(t)) &\cong \begin{cases} \underline{\operatorname{Ext}}^q_A(X, Y)_{t-s} & \text{if } 0 \leq q \leq n-3,\\ D(\underline{\operatorname{Hom}}_A(Y, X_\nu)_{s-t-n+2}) & \text{if } q = n-2,\end{cases}
\end{aligned}$$

by [24, Lemma 2.3] (cf. the proof of [24, Lemma 3.13]). We also use the following facts:

(i) If $i \leq 0$, then $\underline{\operatorname{Hom}}_A(X, A)_i = 0$ ([24, Lemma 3.7]).
(ii) If $i < 0$, or $i \leq 0$ and $X \not\cong Y$, then $\underline{\operatorname{Hom}}_A(X, Y)_i = 0$ ([24, Lemma 3.9]).
(iii) If $q \geq 1$ and $i \neq -q$, then $\underline{\operatorname{Ext}}^q_A(X, Y)_i = 0$ ([24, Lemma 3.8]).

We will now show that $\operatorname{Ext}^q_{\mathscr{C}}(E_i, E_j) = 0$ for every $q$ when $0 < i - j < \ell$. It is enough to consider the following cases where $A \not\cong X, Y \in \operatorname{Ind}^0(\operatorname{CM}^{\mathbb{Z}}(A))$.

(1) The case $E_i = \mathcal{A}(s), E_j = \mathcal{A}(t)$ for some $s, t \in \mathbb{Z}$: If $0 < i - j < \ell$, then $0 < s - t < n - 2$. Since $t - s < 0$ and $s - t - n + 2 < 0$, we have $\operatorname{Ext}^q_{\mathcal{A}}(\mathcal{A}(s), \mathcal{A}(t)) = 0$ for every $q \in \mathbb{Z}$.

(2) The case $E_i = \mathcal{A}(s), E_j = \mathcal{X}(t)$ for some $s, t \in \mathbb{Z}$: If $j < i < j + \ell$, then $E_i = \mathcal{A}(s), E_j = \mathcal{X}(t)$ are positioned as follows

$$\mathcal{X}_{\sigma^{-r}(1)}(r(n-2)), \dots \mathcal{X}(t) = \mathcal{X}(r(n-2)), \dots, \mathcal{X}_{\sigma^{-r}(\alpha)}(r(n-2)),$$
$$\mathcal{A}(r(n-2)+1), \dots, \mathcal{A}(s), \dots, \mathcal{A}((r+1)(n-2)),$$

that is, $t = r(n-2)$ and $r(n-2)+1 \le s \le (r+1)(n-2)$ for some $r \in \mathbb{Z}$, so $t < s \le t+n-2$. Since $t-s < 0$ and $s-t-n+2 \le 0$, we have $\operatorname{Ext}^q_{\mathcal{A}}(\mathcal{A}(s), \mathcal{X}(t)) = 0$ for every $q \in \mathbb{Z}$ by (i).

(3) The case $E_i = \mathcal{X}(s), E_j = \mathcal{A}(t)$ for some $s, t \in \mathbb{Z}$: If $j < i < j + \ell$, then $E_i = \mathcal{X}(s), E_j = \mathcal{A}(t)$ are positioned as follows

$$\mathcal{A}(r(n-2)-n+3), \dots, \mathcal{A}(t), \dots, \mathcal{A}(r(n-2)),$$
$$\mathcal{X}_{\sigma^{-r}(1)}(r(n-2)), \dots \mathcal{X}(s) = \mathcal{X}(r(n-2)), \dots, \mathcal{X}_{\sigma^{-r}(\alpha)}(r(n-2)),$$

that is, $r(n-2)-n+3 \le t \le r(n-2)$ and $s = r(n-2)$ for some $r \in \mathbb{Z}$, so $t \le s < t+n-2$. Since $t-s \le 0$ and $s-t-n+2 < 0$, we have $\operatorname{Ext}^q_{\mathcal{A}}(\mathcal{X}(s), \mathcal{A}(t)) = 0$ for every $q \in \mathbb{Z}$ by (i).

(4) The case $E_i = \mathcal{X}(s), E_j = \mathcal{Y}(t)$ for some $s, t \in \mathbb{Z}$: If $j < i < j + \ell$, then $E_i = \mathcal{X}(s), E_j = \mathcal{Y}(t)$ are positioned as either

$$\mathcal{X}_{\sigma^{-r}(1)}(r(n-2)), \dots, \mathcal{Y}(t) = \mathcal{Y}(r(n-2)), \dots, \mathcal{X}(s) = \mathcal{X}(r(n-2)), \dots, \mathcal{X}_{\sigma^{-r}(\alpha)}(r(n-2))$$

or

$$\mathcal{X}_{\sigma^{-r}(1)}(r(n-2)), \dots, \mathcal{X}_\nu(r(n-2)), \cdots, \mathcal{Y}(t) = \mathcal{Y}(r(n-2)), \dots, \mathcal{X}_{\sigma^{-r}(\alpha)}(r(n-2)),$$
$$\mathcal{A}(r(n-2)+1), \dots, \mathcal{A}((r+1)(n-2)-1), \mathcal{A}((r+1)(n-2)),$$
$$\mathcal{X}_{\sigma^{-(r+1)}(1)}((r+1)(n-2)), \dots, \mathcal{X}(s) = \mathcal{X}((r+1)(n-2)), \dots, \mathcal{X}_{\sigma^{-(r+1)}(\alpha)}((r+1)(n-2)),$$

so either $s = t$ and $Y \not\cong X$, or $s = t+n-2$ and $Y \not\cong X \otimes_A \omega_A(n-2) \cong X_\nu$.

(a) The case $s = t$ and $Y \not\cong X$: If $0 \le q \le n-3$, then

$$\operatorname{Ext}^q_{\mathcal{A}}(\mathcal{X}(s), \mathcal{Y}(t)) \cong \underline{\operatorname{Ext}}^q_A(X, Y)_0 = 0$$

by (ii) and (iii) since $Y \not\cong X$. If $q = n-2$, then $\operatorname{Ext}^{n-2}_{\mathcal{A}}(\mathcal{X}(s), \mathcal{Y}(t)) \cong D(\underline{\operatorname{Hom}}_A(Y, X_\nu)_{s-t-n+2}) = 0$ by (ii) since $s-t-n+2 = -n+2 < 0$.

(b) The case $s = t+n-2$ and $Y \not\cong X_\nu$: If $0 \le q \le n-3$, then $\operatorname{Ext}^q_{\mathcal{A}}(\mathcal{X}(s), \mathcal{Y}(t)) \cong \underline{\operatorname{Ext}}^q_A(X, Y)_{t-s} = 0$ by (ii) and (iii) since $t-s = -n+2 < 0$ and $t-s = -n+2 \ne -q$. If $q = n-2$, then $\operatorname{Ext}^{n-2}_{\mathcal{A}}(\mathcal{X}(s), \mathcal{Y}(t)) \cong D(\underline{\operatorname{Hom}}_A(Y, X_\nu)_0) = 0$ by (ii) since $Y \not\cong X_\nu$.

It follows that $\{E_i\}_{i \in \mathbb{Z}}$ satisfies (H3).

On the other hand, if $i \le j$ and $E_i = \mathcal{X}(s), E_j = \mathcal{Y}(t)$ for some $X, Y \in \operatorname{Ind}^0(\operatorname{CM}^{\mathbb{Z}}(A))$ and $s, t \in \mathbb{Z}$, then $s \le t$, so $t - s \ge 0$ and $s-t-n+2 < 0$, hence $\operatorname{Ext}^q_{\mathcal{A}}(\mathcal{X}(s), \mathcal{Y}(t)) = 0$ for every $q \ne 0$ by (i), (ii), and (iii). It follows that $\{E_i\}_{i \in \mathbb{Z}}$ is a geometric helix of period $\ell$ for $\mathscr{D}^b(\operatorname{tails} A)$.

Since $E_0, \dots, E_{\ell-1}$ is full by [24, Lemma 3.12], $\{E_i\}_{i \in \mathbb{Z}}$ is full by [15, Lemma 3.16, Remark 3.17].

It follows that $C := C(\operatorname{tails} A, \{E_i\}_{i\in\mathbb{Z}})$ is a right noetherian AS-regular $\mathbb{Z}$-algebra of dimension $n-1$ such that $\operatorname{tails} A \cong \operatorname{tails} C$ by Theorem 6.4 and Theorem 6.5.

If $Q \subset \mathbb{P}^{n-1}$ is a (commutative) smooth quadric hypersurface, then

$$Q \cong \operatorname{Proj} k[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_n^2),$$

so

$$\operatorname{coh} Q \cong \operatorname{tails} k[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_n^2),$$

hence the final assertion (cf. [24, Theorem 1.1]). □

*Remark* 6.15.
(1) Presumably, the AS-regular $\mathbb{Z}$-algebra $C$ constructed above is not 1-periodic (see the quiver presentations of $\operatorname{End}_{\mathcal{A}}(\oplus_{i=0}^{\ell-1} E_i)$ in [24, Section 3.5]), so that $C$ is not a $\mathbb{Z}$-algebra associated to any (AS-regular) graded algebra by Lemma 2.20.
(2) Let $Q \subset \mathbb{P}^{n-1}$ be a (commutative) smooth quadric hypersurface. If $n$ is odd, then the final assertion of the above theorem follows from [3, Proposition 3.3]. If $n$ is even, then the $\mathbb{Z}$-algebra $C$ constructed in the above theorem is AS-regular, but not Koszul.

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