An Abstract Characterization of Noncommutative Projective Spaces (w/ Izuru Mori)

Adam Nyman

Western Washington University

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Conventions

- always work over a field k
- always work with right modules
- always let C denote a (k-linear) abelian category.

Noncommutative cohproj (Polishchuk (2005))

Motivation: Serre's Theorem

If A is commutative fg in degree 1 and X is the projective scheme associated to A, then $\text{cohproj}A \equiv \text{coh}X$.

Let A be connected coherent $\mathbb{Z}$-graded algebra.

Definition

- $\text{coh}A =$ cat. of (graded right) coherent modules
- $\text{tors}A =$ full subcat. of right-bounded modules.
- $\text{cohproj}A := \text{coh}A/\text{tors}A$

Goal

Main Theorem

We describe necessary and sufficient conditions on C so that

$$\mathsf{C} \equiv \mathsf{cohproj} A$$

where A is coherent, AS-regular and compatibly periodic.

Application

Mori-Ueyama show *standard* smooth quadrics of S. P. Smith and M. Van den Bergh satisfy our criteria. It will follow that they are noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$'s.

Part 1

Noncommutative Quadrics

The Noncommutative Quadrics of Smith and Van den Bergh (2013)

Let S =connected noetherian $\mathbb{Z}$-graded ring.

Noncommutative Quadrics

Categories cohproj$S/(z)$ with

- S Gorenstein, Koszul, and Hilbert series $= (1-t)^{-4}$, and
- $z \in S_2$ is central and $S/(z)$ is a domain.

Intuition

cohproj$S = \text{nc}\mathbb{P}^3$ and $z = 0$ is nc quadric hypersurface.

Using deformation theory, M. Van den Bergh proved there should be other noncommutative quadrics!

Part 2

$\mathbb{Z}$-Algebras and Noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$'s

$\mathbb{Z}$-algebras (Bondal and Polishchuk (1993))

A $\mathbb{Z}$-algebra is ring A with vector space decomposition

$$\bigoplus_{i,j\in\mathbb{Z}} A_{ij}$$

such that

- $A_{ij}A_{jk} \subset A_{ik}$,
- $A_{ij}A_{kl} = 0$ for $k \neq j$, and
- A_{ii} contains a unit e_i so that $e_i A = \bigoplus_j A_{ij}$.

A does not have unity and is not a domain.

Periodicity (Van den Bergh (2011))

Periodic $\mathbb{Z}$-algebras generalize $\mathbb{Z}$-graded algebras

Let A be a $\mathbb{Z}$-algebra. Let $A(\ell)$ be the $\mathbb{Z}$-algebra with

$$A(\ell)_{ij} := A_{i+\ell,j+\ell}$$

and with multiplication inherited from A.

Definition

A is ℓ-periodic if $A \cong A(\ell)$ as algebras.

Observation

If B is $\mathbb{Z}$-graded, B is Morita equivalent to a 1-periodic $\mathbb{Z}$-algebra.

Definition of AS-Regularity

Let A be a connected $\mathbb{Z}$-algebra.

A is AS-regular of dimension d and Gorenstein parameter ℓ if

1. $\operatorname{pd} e_i(A/A_{\geq 1}) = d$ for all $i \in \mathbb{Z}$, and
2. $\operatorname{Ext}^q(e_i(A/A_{\geq 1}), e_j A) \cong \begin{cases} k & \text{if } q = d, j = i - \ell \\ 0 & \text{otherwise.} \end{cases}$

Theorem (N (2020))

A is a 2-periodic AS-regular of dim 2 and Gorenstein parameter 2, f.g. in degree 1 iff $A \cong \mathbb{S}^{nc}(A_{01})$.

The AS-regular $\mathbb{Z}$-algebras of dimension 3 with polynomial growth have yet to be classified. Periodicity should be a key hypothesis.

Noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$ (Van den Bergh (2011))

Definition

A noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$ is a category of the form $\text{cohproj} A$ where

- A has polynomial growth and
- A is AS-regular of dimension 3 and Gorenstein parameter 4.

Theorem (Van den Bergh (2011))

Every noncommutative deformation of $\mathbb{P}^1 \times \mathbb{P}^1$ is a noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$.

Part 3

Compatible Periodicity

Compatible Periodicity I

A AS-regular of dimension d and Gorenstein parameter ℓ.

Theorem (Mori-N (2020))

Let D denote duality with respect to k. There is an isomorphism

$$\lim_{n\to\infty} \underline{\mathrm{Ext}}^d(A/A_{\geq n}, e_i A) \longrightarrow D(Ae_{i+\ell}).$$

Definition

If A is ℓ-periodic with periodicity $\psi : A \to A(\ell)$, then A is compatibly ℓ-periodic if for all $a_{ij} \in A_{ij}$

$$\begin{array}{ccc} \lim_{n\to\infty} \underline{\mathrm{Ext}}^d(A/A_{\geq n}, e_j A) & \longrightarrow & D(Ae_{j+\ell}) \\ \text{left-mult by } a_{ij} \downarrow & & \downarrow \text{rt-mult by } \psi(a_{ij}) \\ \lim_{n\to\infty} \underline{\mathrm{Ext}}^d(A/A_{\geq n}, e_i A) & \longrightarrow & D(Ae_{i+\ell}) \end{array}$$

commutes.

Compatible Periodicity II

Theorem (Mori-N (2020))

If B is $\mathbb{Z}$-graded coherent AS-regular algebra of Gorenstein parameter ℓ, then there exists a $\mathbb{Z}$-algebra A which is

- compatibly ℓ-periodic
- coherent
- AS-regular of Gorenstein parameter ℓ

such that

$$\text{cohproj} B \equiv \text{cohproj} A.$$

Part 4

Geometric Helices

Exceptional Sequences (Bondal and Polishchuk (1993))

Let T be triangulated, $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ objects in T.

Definition

The sequence $(\mathcal{E}_1, \ldots, \mathcal{E}_n)$ is

- full if $\langle \mathcal{E}_1, \ldots, \mathcal{E}_n \rangle = \mathrm{T}$.
- exceptional of length n if
  1. $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_i[q]) = \begin{cases} k & \text{for } q = 0 \\ 0 & \text{otherwise.} \end{cases}$,
  2. $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j[q]) = 0$ if $i > j$.

Examples

1. $\mathrm{T} = D^b(\mathrm{coh}\mathbb{P}^n)$, $(\mathcal{O}_{\mathbb{P}^n}(a), \mathcal{O}_{\mathbb{P}^n}(a+1), \ldots, \mathcal{O}_{\mathbb{P}^n}(a+n))$ is full and exceptional.
2. Kapranov proves $\mathrm{T} = D^b(\mathrm{coh}\mathbb{P}^1 \times \mathbb{P}^1)$, has full exceptional sequences of length 4.

Canonical Bimodules (Mori-Ueyama (2019))

Definition

A canonical bimodule of C is an autoequivalence $- \otimes \omega_C$ such that for some n

$$- \otimes^{\mathbf{L}} \omega_C[n]$$

is a Serre functor on $D^b(C)$.

Motivation

If X is a smooth projective variety, then tensoring with the canonical sheaf is the canonical bimodule.

Definition of Geometric Helices (Mori-Ueyama(2019))

Suppose C has a canonical bimodule ω_{C}.

Definition

A geometric helix of period ℓ is a sequence of objects $(\mathcal{E}_i)_{i\in\mathbb{Z}}$ in $D^b(\mathsf{C})$ such that for every i

- $(\mathcal{E}_i, \ldots, \mathcal{E}_{i+\ell-1})$ is exceptional and full,
- $\mathcal{E}_{i+\ell} \otimes^{\mathsf{L}} \omega_{\mathsf{C}} \cong \mathcal{E}_i$
- $(\mathcal{E}_i)_{i\in\mathbb{Z}}$ is *geometric*, i.e. $\mathrm{Hom}(\mathcal{E}_i, \mathcal{E}_j[q]) = 0$ for $q \neq 0$ and $i \leq j$.

Part 5

The Main Theorem

Statement of the Main Theorem

Theorem (Mori-N (2020))

There is an equivalence $\mathsf{C} \equiv \text{cohproj}A$ where A is

- coherent,
- compatibly ℓ-periodic, and
- AS-regular $\mathbb{Z}$-algebra of dimension $\text{gldim}\,\mathsf{C} + 1$ and Gorenstein parameter ℓ

if and only if

C has a canonical bimodule and a geometric helix of period ℓ.

- Generalizes the $\mathbb{Z}$-graded version of Mori and Ueyama.
- Main technical challenge: to prove AS-regular coherent $\mathbb{Z}$-algebras have canonical bimodule, we must establish local duality for $\mathbb{Z}$-algebras

Application to Noncommutative Quadrics (Mori-N (2020))

Mori-Ueyama prove (standard) noncommutative quadrics of Smith and Van den Bergh have

- a canonical bimodule
- and a geometric helix of period 4

Main Theorem $\Rightarrow$ such spaces are of the form cohproj$A$ where A is AS-regular of dimension 3 and Gorenstein parameter 4. We observe A has polynomial growth. Thus cohproj$A$ is a noncommutative $\mathbb{P}^1 \times \mathbb{P}^1$'s.

Thank you!